

# INTEGER CONJUGACY CLASSES OF $SL(3, \mathbb{Z})$ AND HESSENBERG MATRICES.

OLEG KARPENKOV

**ABSTRACT.** In this paper we study the problem of description of conjugacy classes in the group  $SL(n, \mathbb{Z})$ . We expand Gauss Reduction Theory that gives the answer for the case  $n = 2$  to the multidimensional case. Reduced Hessenberg matrices now play the role of reduced matrices. For the case of three-dimensional matrices having a real and two complex-conjugate eigenvalues we show that perfect Hessenberg matrices distinguish conjugacy classes asymptotically. An important tool used in our approach is to determine minima of Markoff-Davenport characteristics at the vertices of Klein-Voronoi continued fractions. We conclude the paper with several open questions arising here.

## CONTENTS

1. Introduction	2
2. Basic properties of Hessenberg matrices	4
2.1. Reduction to Hessenberg matrices	5
2.2. On identification of Hessenberg matrix	6
2.3. A family of Hessenberg matrices with given Hessenberg type	7
3. Dirichlet orbits and Markoff-Davenport characteristics	9
3.1. Dirichlet orbits	9
3.2. Markoff-Davenport characteristic	10
3.3. Homogeneous forms associated to $SL(n, \mathbb{Z})$ -operators	11
4. Multidimensional continued fractions in the sense of Klein-Voronoi	12
4.1. General definitions	12
4.2. Algebraic continued fractions	13
4.3. Fundamental domains of sails and reduced Hessenberg matrices	14
5. Three-dimensional NRS-matrices	15
5.1. A few words about two-dimensional case	16
5.2. Asymptotic structure of the subset of perfect Hessenberg NRS-matrices	17
5.3. Asymptotic behaviour of Klein-Voronoi continued fractions for the NRS-case	21
5.4. Examples of NRS-matrices for a given Hessenberg type	27
5.5. Two examples of couples of Hessenberg matrices	29
6. Open problems and questions	30

---

*Date:* 15 September 2009.

*Key words and phrases.* Integer matrix reduction theory, Hessenberg matrices, Markoff-Davenport characteristic, Klein-Voronoi continued fractions, convex hulls.

The work is partially supported by RFBR SS-709.2008.1 grant and by FWF grant No. S09209.

6.1. Three dimensional NRS-matrices	30
6.2. Three dimensional RS-matrices	31
6.3. Four-dimensional case	33
References	35

## 1. INTRODUCTION

In this paper we introduce and investigate a natural structure of the set of conjugacy classes in  $SL(n, \mathbb{Z})$ . Gauss Reduction Theory provides a solution for the simplest case  $n = 2$  (see, for instance, in [20], and [14]). We write down a multidimensional analog of Gauss reduction an focus on the following question: *how good the set of reduced operators "approximate" the set of conjugacy classes?* We study this question in the simplest case of  $SL(3, \mathbb{Z})$ -operators with irreducible characteristic polynomial over rational numbers having one real and two complex roots. In particular we obtain that perfect Hessenberg matrices distinguish conjugacy classes asymptotically (Theorem 5.11 below).

According to Gauss Reduction Theory any conjugacy class of  $SL(2, \mathbb{Z})$ -operators with two real eigenvalues defines a special periodic ordinary continued fraction whose period up to cyclic permutations is a complete invariant of the conjugacy class (see in Subsection 5.1). Moreover Gauss reduction algorithm applied to any operator of the conjugacy class gives a certain *reduced* operator. The number of reduced operators in a conjugacy class equals to the length of the period of the mentioned above ordinary continued fraction.

We expand to the multidimensional case ( $n \geq 3$ ) the idea to use continued fractions and their periods as a main tool to construct complete invariants. We put together old constrictions of Klein's polyhedron (introduced in [15], [16]) and Voronoi's continued fraction (described in [23]) and the ideas of J. A. Buchmann (see in [3]) to obtain a general definition of geometric Klein-Voronoi multidimensional continued fraction. In our study of  $SL(3, \mathbb{Z})$  conjugacy classes we work with periods of Klein-Voronoi continued fractions. We use *upper Hessenberg matrices* as a multidimensional analog of reduced operators.

Notice that the structure of periods of multidimensional Klein-Voronoi continued fractions is not clearly known as it is in the classical case. The first attempts to study the periods were made by V. I. Arnold [1], E. I. Korkina [18], and author [11] and [12]. Nevertheless several characteristics of such periods can be taken as useful invariants. In particular we take the values of Markoff-Davenport characteristic on the vertices of Klein-Voronoi continued fractions to distinguish  $SL(3, \mathbb{Z})$  operators.

**Integer notation.** A point (vector) is said to be *integer* if all its coordinates are integers. A segment is said to be *integer* if its endpoints are integer. An *integer length* of an integer segment is the number of integer points contained in the interior of the segment plus one. An  $m$ -dimensional plane is said to be *integer* if the integer vectors contained in the plane generate the Abelian group of rank  $m$ . A linear transformation is said to be *integer* if it preserves lattice of integer points. Two matrices are said to be *integer*

*conjugate* if they are conjugate and the transformation matrix corresponds to an integer linear transformation.

**Hessenberg matrices and Hessenberg complexity.** A matrix  $M$  in the group  $SL(n, \mathbb{R})$  of the form

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n-2} & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n-2} & a_{2,n-1} & a_{2,n} \\ 0 & a_{3,2} & \cdots & a_{3,n-2} & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & \cdots & 0 & a_{n,n-1} & a_{n,n} \end{pmatrix}$$

is called an (*upper*) *Hessenberg* matrix (such matrices were first studied by K. Hessenberg in [10]). The integer

$$\prod_{j=1}^{n-1} |a_{j+1,j}|^{n-j}$$

is called the *Hessenberg complexity* of the matrix  $M$  and denoted by  $\varsigma(M)$ .

Further, we use the following notation. Let  $M = (a_{i,j})$  be a Hessenberg matrix with positive Hessenberg complexity. We say that the matrix  $M$  is of *Hessenberg type*

$$\langle a_{1,1}, a_{1,2} | a_{2,1}, a_{2,2}, a_{2,3} | \cdots | a_{n-1,1}, a_{n-1,2}, \dots, a_{n-1,n} \rangle.$$

**Definition 1.1.** An integer Hessenberg matrix in  $SL(n, \mathbb{Z})$  with irreducible characteristic polynomial is said to be *perfect* if for any couple of integers  $(i, j)$  satisfying  $1 \leq i < j+1 \leq n$  the following inequalities hold:  $0 \leq a_{i,j} < a_{j+1,j}$ .

In other words all elements of all the columns except the last column of a perfect Hessenberg matrix are nonnegative integers, the maximal elements in these columns are the lowest nonzero ones (i.e.,  $a_{j+1,j}$ ,  $j = 1, \dots, n-1$ ).

Note that the set of values of Hessenberg complexity for perfect Hessenberg matrices of  $SL(n, \mathbb{Z})$  for  $n \geq 2$  coincides with the set of positive integers. An integer Hessenberg matrix has the unit complexity iff it  $a_{2,1} = \cdots = a_{n,n-1} = 1$ , such matrices are called *Frobenius* matrices. The elements of the last column of a Frobenius matrix are the coefficients of the characteristic polynomial multiplied alternatively by  $\pm 1$ .

**Reduced Hessenberg matrices.** In this paper we mostly study matrices with irreducible characteristic polynomials over rational numbers. Any such matrix is conjugate to a perfect Hessenberg matrix with positive Hessenberg complexity (see Theorem 2.1 below). Actually, there are infinitely many perfect Hessenberg matrices integer conjugate to a given one.

**Definition 1.2.** A perfect Hessenberg matrix  $W$  is said to be *reduced* if for any perfect Hessenberg matrix  $M$  integer conjugate to  $W$  we have

$$\varsigma(W) \leq \varsigma(M).$$

Otherwise we say that the matrix is *nonreduced*.

Note that it is possible to have several (but always finitely many) reduced perfect Hessenberg matrices conjugate to each other, see Example 5.22. In this case such matrices are of the same Hessenberg complexity. This happens also for matrices in  $SL(2, \mathbb{Z})$ .

**Families of  $SL(3, \mathbb{Z})$ -operators with irreducible characteristic polynomials and two complex conjugate eigenvalues.** We call a matrix with all real eigenvalues (a real and two complex conjugate eigenvalues) a *real spectra matrix* (*non-real spectra matrix*) or an *RS-matrix* (*NRS-matrix*), for short.

It is easy to show that any NRS-matrix is conjugate to infinitely many Hessenberg matrices. Nevertheless, experiments show that *for any fixed Hessenberg type there exist only finitely many NRS-matrices that are not perfect*. The similar statement does not hold for RS-matrices. These allow us to conclude that the results of the experiments are really unexpected.

So, let us focus on families of  $SL(3, \mathbb{Z})$ -matrices with the same Hessenberg type. From Corollary 2.7 it follows that any such family forms a two-dimensional lattice in a two-dimensional affine plane in the nine-dimensional space of all matrices. Further in Theorem 5.7 we state that *the set of NRS-matrices in such plane coincides with the set of lattice points of the union of two convex hulls of parabolas except for some points in the neighborhood of parabolas* (as on Figures 1, 3–6).

Hence, the set of NRS-matrices for a given Hessenberg type has two asymptotic directions corresponding to the parabolas. In Theorem 5.11 we prove that *for the sequence of  $SL(3, \mathbb{Z})$ -matrices of any ray with an asymptotic direction we have only reduced matrices starting from certain moment*.

**Description of the paper.** In the next section we study basic properties of Hessenberg matrices of arbitrary dimensions. In particular, we discuss existence and finiteness of reduced Hessenberg operators integer conjugate to a given one. Further we investigate families of Hessenberg operators with given Hessenberg type. In Section 3 we study the Markoff-Davenport characteristics of Dirichlet orbits (i.e. orbits for the action of the Dirichlet group, see also in [2]) and their relations with the Hessenberg complexity. Further in Section 4 we define multidimensional Klein-Voronoi continued fractions, that are considered to be a multidimensional analog of ordinary continued fractions in Gauss Reduction Theory. Section 5 contains main results for the case of NRS-matrices in  $SL(3, \mathbb{Z})$ . Finally, in Section 6 we formulate several questions for further study related to RS- and NRS-matrices in  $SL(3, \mathbb{Z})$  and say a few words about four dimensional case.

**Acknowledgment.** The author is grateful to V. I. Arnold for constant attention to this work, to H. W. Lenstra and E. I. Pavlovskaya for useful remarks, and University of Leiden and Technical University of Graz for the hospitality and excellent working conditions.

## 2. BASIC PROPERTIES OF HESSENBERG MATRICES

In this section we study some basic properties of Hessenberg matrices. First, we show in Subsections 2.1 and 2.2 that for any  $SL(n, \mathbb{Z})$ -matrix with irreducible characteristic

polynomial there exists at least one reduced Hessenberg matrix integer conjugate to the given one. In the majority of the observed examples such matrix is unique. Nevertheless, in some cases that is not true. In Subsections 2.2 we prove that a Hessenberg matrix is uniquely defined by its characteristic polynomial and Hessenberg type. This implies that the number of reduced Hessenberg matrices conjugate to a given one is always finite. In Subsection 2.3 we describe the lattice structure of the family of Hessenberg matrices with a given Hessenberg type.

### 2.1. Reduction to Hessenberg matrices.

**Theorem 2.1.** *Any matrix in  $SL(n, \mathbb{Z})$  with irreducible characteristic polynomial is integer conjugate to a reduced Hessenberg matrix with positive Hessenberg complexity.*

We start with the following lemma.

**Lemma 2.2.** *Any matrix in  $SL(n, \mathbb{Z})$  with irreducible characteristic polynomial is integer conjugate to a Hessenberg matrix with positive Hessenberg complexity.*

*Proof.* Let  $M$  be a matrix in  $SL(n, \mathbb{Z})$  with irreducible characteristic polynomial, and  $A$  be a linear operator with matrix  $M$  in some fixed integer basis. Take any integer vector  $v$  of unit integer length and consider a set of vector spaces

$$V_i = \text{Span}(v, A(v), A^2(v), \dots, A^{i-1}(v)),$$

for  $i = 1, \dots, n$  (here we denote the span of vectors  $v_1, \dots, v_m$  by  $\text{Span}(v_1, \dots, v_m)$ ). Since for any integer  $j$  the vector  $A^j(v)$  is integer, the spaces  $V_i$  are integer. Since the characteristic polynomial of  $A$  is irreducible, the dimension of  $V_i$  equals  $i$  and the set of all spaces  $V_i$  forms a complete flag in  $\mathbb{R}^n$ .

Let us describe the following basis of  $\mathbb{R}^n$ , we denote it by  $\{g_i\}$ . We choose  $v$  as  $g_1$ . For any  $i > 1$  we chose an integer vector  $g_i \in V_i$  such that  $g_i$  together with all vectors of  $V_{i-1}$  generate an integer lattice in  $V_i$ . Notice that the choice of  $g_i$  is not unique. The vectors  $g_1, \dots, g_n$  form a basis of  $\mathbb{R}^n = V_n$ . Let  $\hat{M} = (\hat{a}_{i,j})$  be the matrix of the operator  $A$  in the basis  $\{g_i\}$ . By construction, this matrix is Hessenberg.

Since the basis vectors  $g_1, \dots, g_n$  generate the integer lattice, the matrix  $\hat{M}$  is integer conjugate to the matrix  $M$ .

Since the characteristic polynomial of  $A$  is irreducible, the integer spaces  $V_i$  are not invariant subspaces of  $A$ . Hence, the integers  $\hat{a}_{i+1,i}$  are non-zero for  $i = 1, \dots, n-1$ . Therefore, the Hessenberg complexity of  $\hat{M}$  is positive. This concludes the proof of Proposition 2.2.  $\square$

*Proof of Theorem 2.1.* By Lemma 2.2 it is enough to prove the theorem for a Hessenberg matrix with positive Hessenberg complexity. Let  $M = (a_{i,j})$  in  $SL(n, \mathbb{Z})$  be such matrix. Suppose also, that  $A$  is an operator with matrix  $M$  in some integer basis  $\{e_i\}$ . Now we construct inductively a reduced Hessenberg matrix that is conjugate to  $M$ .

We put  $\hat{e}_1$  to be equal to  $e_1$ .

Choose  $c_{1,1}$  and  $\hat{a}_{1,1}$  such that

$$A(\hat{e}_1) = |a_{2,1}|(\text{sign}(a_{2,1})e_2 + c_{1,1}\hat{e}_1) + \hat{a}_{1,1}e_1 \quad \text{and} \quad 0 \leq \hat{a}_{1,1} < |a_{2,1}|.$$

Let

$$\hat{e}_2 = \text{sign}(a_{2,1})e_2 + c_{1,1}\hat{e}_1.$$

Suppose, that for some  $k$  we have constructed  $\hat{e}_i$  for all  $i \leq k$  and  $\hat{a}_{i,j}$  for all  $i \leq j \leq k-1$ . Let us construct  $\hat{e}_{k+1}$  and  $\hat{a}_{i,k}$  for  $i = 1, \dots, k$ . Choose  $c_{i,k}$  and  $\hat{a}_{i,k}$  for  $i = 1, \dots, k$  such that

$$A(\hat{e}_k) = |a_{k+1,k}| \left( \text{sign}(a_{k+1,k})e_{k+1} + \sum_{i=1}^k c_{i,k}\hat{e}_i \right)$$

and  $0 \leq \hat{a}_{i,k} < a_{k+1,k}$  for  $i = 1, \dots, k$ . We put

$$\hat{e}_{k+1} = \text{sign}(a_{k+1,k})e_{k+1} + \sum_{i=1}^k c_{i,k}\hat{e}_i$$

and calculate the coefficients  $\hat{a}_{i,k}$  from the expression for  $A(\hat{e}_k)$  in the system of linearly independent vectors  $\hat{e}_1, \dots, \hat{e}_{k+1}$ .

The matrix  $\hat{M}$  of the operator  $A$  in the basis  $\{\hat{e}_i\}$  is of Hessenberg type

$$\left\langle \hat{a}_{1,1}, |a_{2,1}| \left| \hat{a}_{1,2}, \hat{a}_{2,2}, |a_{3,2}| \right| \dots \left| \hat{a}_{1,n-1}, \dots, \hat{a}_{n-1,n-1}, |a_{n,n-1}| \right\rangle.$$

By the definition,  $\hat{M}$  is a perfect Hessenberg matrix.

From the construction it follows that the matrices  $M$  and  $\hat{M}$  are integer conjugate. So there exists at least one perfect Hessenberg matrix integer conjugate to  $M$ . Since the set of values of Hessenberg complexity is discrete and bounded from below, there exist a reduced Hessenberg matrix among the perfect Hessenberg matrices integer conjugate to  $M$ .  $\square$

**Corollary 2.3.** *Let  $A$  be an  $SL(n, \mathbb{Z})$ -operator and  $v$  be any integer vector of unit integer length. Then there exists a unique integer basis  $\{g_i\}$  such that*

- 1)  $g_1 = v$ ;
- 2)  $g_i \in V_i$  where  $V_i = \text{Span}(v, A(v), A^2(v), \dots, A^{i-1}(v))$ ;
- 3) the matrix  $M$  of the operator  $A$  in the basis  $\{g_i\}$  is perfect Hessenberg.

*Proof.* The statement follows directly from the proof algorithm of Lemma 2.2 and Theorem 2.1.  $\square$

## 2.2. On identification of Hessenberg matrix.

**Proposition 2.4.** *Any Hessenberg matrix with positive Hessenberg complexity is uniquely defined by its Hessenberg type and the characteristic polynomial.*  $\square$

*Proof.* From definition of Hessenberg matrix it is enough to prove the following. The last column in the Hessenberg matrix with positive Hessenberg complexity is uniquely defined by its Hessenberg type and the coefficients of the characteristic polynomial.

Suppose, that we know all the coefficients of the characteristic polynomial and the Hessenberg type for some Hessenberg matrix  $M = (a_{i,j})$ . Then the elements  $a_{i,j}$  for

$1 \leq i \leq n, 1 \leq j \leq n$  are uniquely defined by the Hessenberg type of  $M$ . Let the characteristic polynomial of  $M$  be

$$x^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0.$$

Direct calculations show that for any  $k$  the coefficient  $c_k$  is a polynomial in  $a_{i,j}$  variables that does not depend on  $a_{1,n}, \dots, a_{k,n}$ . The unique monomial for  $c_k$  containing  $a_{k+1,n}$  is

$$\left( \prod_{j=k+1}^{n-1} a_{j+1,j} \right) a_{k+1,n}.$$

Since the Hessenberg complexity of  $M$  is nonzero, we have uniquely defined expressions for  $a_{k+1,n}$  where  $k = n-1, n-2, \dots, 0$ .

So the last column is uniquely defined. This concludes the proof of the proposition.  $\square$

**Corollary 2.5.** *For any  $SL(n, \mathbb{Z})$ -matrix  $M$  with irreducible characteristic polynomial there exist a finitely many reduced Hessenberg matrices integer conjugate to the given one with bounded from above Hessenberg complexity.*

*Proof.* The existence of reduced Hessenberg matrices integer conjugate to  $M$  follows from Theorem 2.1. Let the Hessenberg complexity of that reduced matrix equal  $c$ . Since the conjugate matrices have the same characteristic polynomial and by Proposition 2.4, there exists at most one Hessenberg matrix of a given Hessenberg type conjugate to  $M$ . The number of Hessenberg types with Hessenberg complexity equal to  $c$  is finite. Thus, there is only a finite number of reduced Hessenberg matrices integer conjugate to  $M$ .  $\square$

**2.3. A family of Hessenberg matrices with given Hessenberg type.** We start with two important definitions of integer lattice geometry.

The *integer volume* of a simplex  $\sigma$  with integer vertices is the index of the sublattice generated by the edges of  $\sigma$  in the lattice of all integer vectors in the plane spanned by  $\sigma$ .

Take an integer vector  $v$  and a  $k$ -dimensional plane  $\pi$  containing the integer sublattice of rank  $k$  such that  $v$  is not in  $\pi$ . The *integer distance* from  $v$  to  $\pi$  is the index of the sublattice generated by the integer vectors of the set  $\pi \cup \{v\}$  in the whole integer lattice of the  $(k+1)$ -dimensional plane spanning  $v$  and  $\pi$ .

Consider a vector space  $\mathbb{R}^n$  with an integer basis  $\{e_j\}$ . For a given Hessenberg type

$$\Omega = \langle a_{1,1}, a_{1,2} | a_{2,1}, a_{2,2}, a_{2,3} | \cdots | a_{n-1,1}, \dots, a_{n-1,2}, a_{n-1,n} \rangle$$

we associate vertices  $v_k(\Omega) = (a_{k,1}, \dots, a_{k,k+1}, 0, \dots, 0)$  for  $k = 1, \dots, n-1$ . Let  $O$  be the origin. Denote by  $\sigma(\Omega)$  the  $(n-1)$ -dimensional simplex with vertices  $O, v_1, \dots, v_{n-1}$ .

**Proposition 2.6.** *A Hessenberg matrix  $M$  of type  $\Omega$  with a vector  $v$  in the last column is in  $SL(n, \mathbb{Z})$  iff the following conditions hold (in the integer lattice of the integer basis  $\{e_i\}$ ):*

- i) *the integer volume of  $\sigma(\Omega)$  equals one;*
- ii) *the integer distance from the vector  $v$  in the basis  $\{e_i\}$  to the integer hyperplane containing  $\sigma(\Omega)$  equals one.*

*Proof.* Let  $A$  be an operator with matrix  $M$  in the basis  $\{e_i\}$ .

If  $M$  is in  $SL(n, \mathbb{Z})$ , then  $A$  preserves all integer volumes and integer distances. Since the integer volume of the simplex with vertices

$$O, O+e_1, \dots, O+e_{n-1}$$

equals one, the volume of the image  $\sigma(\Omega)$  equals one. Since the integer distance from the point  $O+e_n$  to the plane spanned by the vectors  $e_1, \dots, e_{n-1}$  equals one, the integer distance from the point  $(\lambda_1, \dots, \lambda_n)$  to the integer hyperplane containing  $\sigma(\Omega)$  also equals one.

Suppose, that conditions i) and ii) hold. Then, the operator  $A$  takes the integer lattice (generated by vectors  $e_1, \dots, e_n$ ) to itself bijectively. Therefore,  $M$  is in  $SL(n, \mathbb{Z})$ .  $\square$

Denote by  $H(\Omega)$  the set of all Hessenberg matrices in  $SL(n, \mathbb{Z})$  of the Hessenberg type  $\Omega$ . For  $k = 1, \dots, n-1$  we denote by  $M_k(\Omega)$  the matrix with zero first  $n-1$  columns and the last one equals to the coordinates of the vector

$$(a_{k,1}, \dots, a_{k,2}, a_{k,k+1}, 0, \dots, 0).$$

**Corollary 2.7.** *Let  $\Omega$  be a Hessenberg type satisfying condition i) of Proposition 2.6, and  $M_0$  be in  $H(\Omega)$ . Then*

$$H(\Omega) = \left\{ M_0 + \sum_{i=1}^{n-1} c_i M_i(\Omega) \mid c_1, \dots, c_n \in \mathbb{Z} \right\}.$$

*Proof.* The statement follows directly from Proposition 2.6.  $\square$

We conclude this subsection with a particular example.

**Example 2.8.** Let us consider matrices of the Hessenberg type  $\langle 0, 1|1, 0, 2 \rangle$ . All matrices of that type form a two-parametric family

$$H_{\langle 0,1|1,0,2 \rangle}^{(1,0,1)}(m, n) = \begin{pmatrix} 0 & 1 & n+1 \\ 1 & 0 & m \\ 0 & 2 & 2n+1 \end{pmatrix}$$

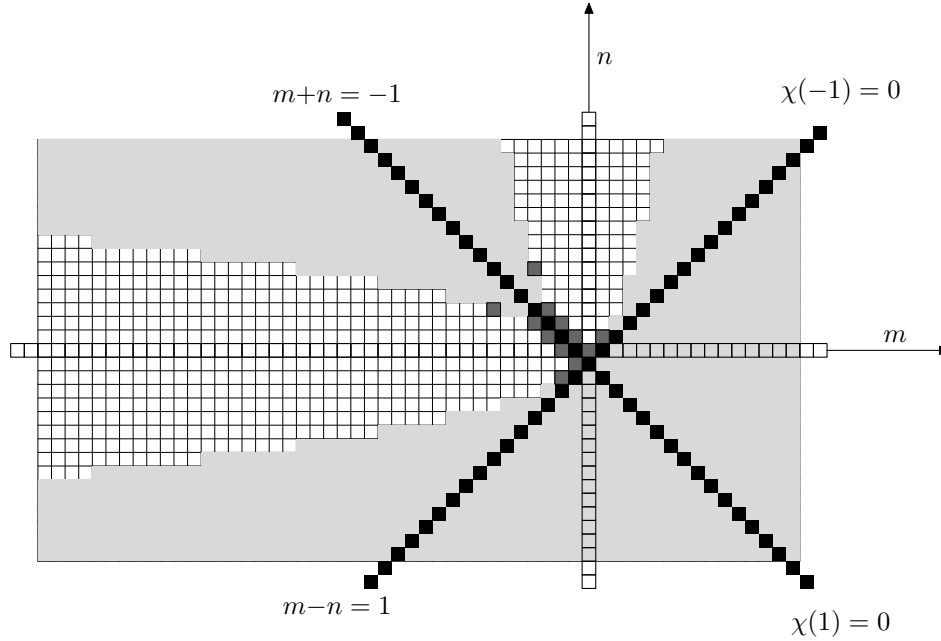
with integer parameters  $m$  and  $n$ . The discriminant of the matrix  $H_{\langle 0,1|1,0,2 \rangle}^{(1,0,1)}(m, n)$  equals  $-44 - 44n^2 - 56mn - 32n^3 + 32m^3 + 16m^2n^2 + 16mn^2 + 16m^2n - 56n - 8m + 52m^2$ .

The set of matrices with negative discriminant for the given family coincides with the union of integer solutions of the following inequalities:

$$2m \leq -n^2 - n - 2 \quad \text{and} \quad 2n \leq m^2 + m.$$

In Figure 1 we show the family of Hessenberg operators of type  $\langle 0, 1|1, 0, 2 \rangle$ . The square in the intersection of the  $m$ -th column and the  $n$ -th row corresponds to the matrix  $H_{\langle 0,1|1,0,2 \rangle}^{(1,0,1)}(m, n)$ . Black squares correspond to the matrices with reducible characteristic




 FIGURE 1. The family of matrices of Hessenberg type  $\langle 0, 1|1, 0, 2 \rangle$ .

polynomials. Light gray squares correspond to the RS-matrices. Dark gray squares correspond to the nonreduced NRS-matrices. Finally, white squares form the set of reduced NRS-Hessenberg matrices.

There are only 12 dark gray squares corresponding to the nonreduced Hessenberg NRS-matrices. We conjecture that there is no other nonreduced NRS-matrices in the family of matrices  $H_{\langle 0, 1|1, 0, 2 \rangle}^{(1, 0, 1)}(m, n)$ .

### 3. DIRICHLET ORBITS AND MARKOFF-DAVENPORT CHARACTERISTICS

The study of Markoff-Davenport characteristic is closely related to the theory of minima of absolute values of homogeneous forms with integer coefficients in  $n$ -variables of degree  $n$ . One of the first works on minima of such forms was written by A. Markoff [22] for the case  $n = 2$  for the forms decomposable into the product of two real linear forms. Further, H. Davenport in series of works [4], [5], [6], [7], and [8] made the first steps for the case of decomposable forms for  $n = 3$ .

In this section we give the definition of Markoff-Davenport characteristic. In Subsection 3.2 we show a relation between values of Markoff-Davenport characteristic and the Hessenberg complexities of Hessenberg matrices of an operator. In Subsection 3.3 we show that Markoff-Davenport characteristic is an absolute value of some homogeneous form of degree equal to the rank of the operator.

**3.1. Dirichlet orbits.** Consider any operator  $A$  of  $SL(n, \mathbb{Z})$  with irreducible characteristic polynomial over the field of rational numbers. The group of all  $SL(n, \mathbb{Z})$ -operators

commuting with  $A$  and having only positive real eigenvalues is called *the Dirichlet group* and denoted by  $\Xi(A)$ . By Dirichlet unity theorem the Dirichlet group  $\Xi(A)$  is homomorphic to some free Abelian group. An orbit of the Dirichlet group consisting of integer points is said to be a *Dirichlet orbit*.

**Example 3.1.** Consider an operator  $A_0$  with matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

The Dirichlet group  $\Xi(A_0)$  is isomorphic to  $\mathbb{Z}^3$  and it is generated by the operators  $A_0^2$ ,  $(E-A_0)^2$ , and  $A_0^3+A_0$ .

**3.2. Markoff-Davenport characteristic.** Let us start with the definition.

**Definition 3.2.** Consider  $A \in SL(n, \mathbb{Z})$ . The *Markoff-Davenport characteristic* (or *MD-characteristic*, for short) of  $A$  is a functional defined on  $\mathbb{R}^n$  as follows: the value at a vector  $v$  is the nonoriented integer volume of the pyramid with vertex at the origin and base with vertices  $v, A(v), \dots, A^{n-1}(v)$ . Denote the MD-characteristic at  $v$  by  $\Delta(A|v)$ .

Further in Subsection 3.3 we show that MD-characteristic is a functional coinciding with the absolute value of some homogeneous form of degree  $n$ .

*Remark 3.3.* The MD-characteristic naturally defines a functional over the set of Dirichlet orbits for the given operator, since the MD-characteristic at any two vectors of the same Dirichlet orbit is the same.

**Proposition 3.4.** Consider an operator  $A$  with Hessenberg matrix  $M$  in some integer basis  $\{e_i\}$ . Then the Hessenberg complexity  $\varsigma(M)$  equals the value of MD-characteristic  $\Delta(A|e_1)$ .  $\square$

*Proof.* Suppose, that the Hessenberg type of the matrix  $M$  is

$$\langle a_{1,1}, a_{1,2} | a_{2,1}, a_{2,2}, a_{2,3} | \dots | a_{n-1,1}, \dots, a_{n-1,2}, a_{n-1,n} \rangle.$$

Denote by  $V_k$  the plane  $\text{Span}(v, A(v), A^2(v), \dots, A^{k-1}(v))$ .

Let us inductively show that  $A^k(e_1) = \left(\prod_{i=1}^k a_{i,i+1}\right) e_{k+1} + v_k$  where  $v_k$  is in  $V_k$ .

We have  $A(e_1) = a_{1,2}e_2 + a_{1,1}e_1$  as the base of induction.

Suppose, that the statement holds for  $k = m$ , i.e.,  $A^m(e_1) = \left(\prod_{i=1}^m a_{i,i+1}\right) e_{m+1} + v_m$  and  $v_m$  is in  $V_m$ . Let us show the statement for  $m+1$ . Since  $M$  is Hessenberg,  $A(v_m)$  is in  $V_{m+1}$ . Therefore, we have

$$\begin{aligned} A^{m+1}(e_1) &= A\left(\left(\prod_{i=1}^m a_{i,i+1}\right) e_{m+1}\right) + A(v_m) = \\ &\left(\prod_{i=1}^{m+1} a_{i,i+1}\right) e_{m+2} + \left(A(v_m) + \left(\prod_{i=1}^m a_{i,i+1}\right) (A(e_{m+1}) - a_{m+1,m+2}e_{m+2})\right). \end{aligned}$$

The second additive in the last expression is in  $V_{m+1}$ . We have proven our statement. Therefore,

$$\Delta(A|e_1) = \prod_{i=1}^{n-1} |a_{i+1,i}|^{n-i} = \varsigma(M).$$

This concludes the proof of the proposition.  $\square$

The proposition implies that for any operator  $A$  and any integer  $d$ , there exist only finitely many Dirichlet orbits with MD-characteristic equal to  $d$ .

**3.3. Homogeneous forms associated to  $SL(n, \mathbb{Z})$ -operators.** Let  $\{e_i\}$  be an integer basis of  $\mathbb{R}^n$ . Consider any  $SL(n, \mathbb{Z})$ -operator  $A$  with irreducible characteristic polynomial. Suppose that it has  $k$  real eigenvalues  $r_1, \dots, r_k$  and  $2l$  complex conjugate eigenvalues  $c_1, \bar{c}_1, \dots, c_l, \bar{c}_l$ , where  $k + 2l = n$ . Let us now define a new basis of vectors  $g_1, \dots, g_{k+2l}$  in the following way. For  $i = 1, \dots, k$  we choose  $g_i$  to be an eigenvector corresponding to the eigenvalue  $r_i$ . For  $j = 1, \dots, l$  we choose  $g_{k+2j-1}$  and  $g_{k+2j}$  to be the real and the imaginary parts of some complex eigenvector corresponding to the eigenvalue  $c_j$ . Consider the system of coordinates

$$OX_1X_2 \dots X_kY_1Z_1Y_2Z_2 \dots Y_lZ_l$$

corresponding to the basis  $\{g_i\}$ .

A form

$$\alpha \left( \prod_{i=1}^k x_i \prod_{j=1}^l (y_j^2 + z_j^2) \right)$$

with nonzero  $\alpha$  is said to be *associated* to the operator  $A$ .

**Theorem 3.5.** *Let  $A$  be an  $SL(n, \mathbb{Z})$ -operator with irreducible characteristic polynomial. Then the MD-characteristic of  $A$  is an absolute value of a form associated to  $A$  for a certain nonzero  $\alpha$ .*

*Proof.* Let us consider the formulas of MD-characteristic of  $A$  in the eigen-basis of vectors

$$g_1, \dots, g_k, g_{k+1} + Ig_{k+2}, g_{k+1} - Ig_{k+2}, \dots, g_{k+2l-1} + Ig_{k+2l}, g_{k+2l-1} - Ig_{k+2l}$$

in  $\mathbb{C}^n$ , where  $I = \sqrt{-1}$ . Let the coordinates in this eigen-basis be  $\{t_i\}$ .

Then for any vector  $v = (t_1, \dots, t_n)$  we have

$$A^j(x) = (r_1^j t_1, \dots, r_k^j t_k, c_1^j t_{k+1}, \bar{c}_1^j t_{k+2}, \dots, c_l^j t_{k+2l-1}, \bar{c}_l^j t_{k+2l}).$$

Therefore,

$$\Delta(A|(t_1, \dots, t_n)) = \alpha \left| \prod_{i=1}^k t_i \prod_{j=1}^l (t_{k+2j-1} t_{k+2j}) \right| = \frac{\alpha}{4^l} \left| \prod_{i=1}^k x_i \prod_{j=1}^l (y_j^2 + z_j^2) \right|$$

Simple calculations show that  $\alpha \neq 0$ .  $\square$

#### 4. MULTIDIMENSIONAL CONTINUED FRACTIONS IN THE SENSE OF KLEIN-VORONOI

Here we describe multidimensional continued fractions in the sense of Klein-Voronoi. We use these continued fractions to find the minimal value of the MD-characteristic for a given operator on the lattice of integer points except the origin.

In 1839 C. Hermite [9] posed the problem of generalizing ordinary continued fractions to the higher-dimensional case. Since then there were many different definitions generalizing different properties of ordinary continued fractions. A nice geometrical generalization of ordinary continued fraction for operators with all real eigenvalues was made by F. Klein in [15] and [16]. We refer to a nice description, properties, and examples of multidimensional continued fractions in the sense of Klein to the books by V. I. Arnold [1] and G. Lachaud [19] and papers by E. I. Korkina [18], M. L. Kontsevich and Yu. M. Suhov [17], and the author [11], [12]. Approximately at the same time of the works by F. Klein G. Voronoi in his dissertation [23] introduced a nice geometric algorithmic definition for all the cases even for operators with couples of complex conjugate eigenvalues. In [3] J. A. Buchmann generalized Voronoi's algorithm making it more convenient for computation of fundamental units in orders.

We use ideas of J. A. Buchmann to define the *multidimensional continued fraction in the sense of Klein-Voronoi* for all the cases. Note that if all the eigenvalues of an operator are real numbers then the Klein's multidimensional continued fraction is a continued fractions in the sense of Klein-Voronoi.

**4.1. General definitions.** Consider an operator  $A$  in  $SL(n, \mathbb{R})$  without multiple eigenvalues. Suppose that it has  $k$  real eigenvalues  $r_1, \dots, r_k$  and  $2l$  complex conjugate eigenvalues  $c_1, \bar{c}_1, \dots, c_l, \bar{c}_l$ , with  $k + 2l = n$ .

Denote by  $T^l(A)$  the set of all real operators commuting with  $A$  such that their real eigenvalues are all unit and the absolute values for all complex eigenvalues equal one. Actually,  $T^l(A)$  is an abelian group with operation of matrix multiplication.

For a vector  $v$  in  $\mathbb{R}^n$  we denote by  $T_A(v)$  the orbit of  $v$  with respect of the action of the group of operators  $T^l(A)$ . If  $v$  is in general position with respect to the operator  $A$  (i.e. it does not lie in invariant planes of  $A$ ), then  $T_A(v)$  is homeomorphic to the  $l$ -dimensional torus. For a vector of an invariant plane of  $A$  the orbit  $T_A(v)$  is also homeomorphic to a torus of positive dimension not greater than  $l$ , or to a point.

For instance, if  $v$  is a real eigenvector, then  $T_A(v) = \{v\}$ . The second example: if  $v$  is in a real hyperplane spanned by two complex conjugate eigenvectors, then  $T_A(v)$  is an ellipse.

As before, let  $g_i$  be a real eigenvector with eigenvalue  $r_i$  for  $i = 1, \dots, k$ ;  $g_{k+2j-1}$  and  $g_{k+2j}$  be vectors corresponding to the real and imaginary parts of some complex eigenvector with eigenvalue  $c_j$  for  $j = 1, \dots, l$ . Again we consider the coordinate system corresponding to the basis  $\{g_i\}$ :

$$OX_1X_2\dots X_kY_1Z_1Y_2Z_2\dots Y_lZ_l.$$

Denote by  $\pi$  the  $(k+l)$ -dimensional plane  $OX_1X_2 \dots X_kY_1Y_2 \dots Y_l$ . Let  $\pi_+$  be the cone in the plane  $\pi$  defined by the equations  $y_i \geq 0$  for  $i = 1, \dots, l$ . For any  $v$  the orbit  $T_A(v)$  intersects the cone  $\pi_+$  in a unique point.

**Definition 4.1.** A point  $p$  in the cone  $\pi_+$  is said to be  $\pi$ -integer if the orbit  $T_A(p)$  contains at least one integer point.

Consider all (real) hyperplanes invariant under the action of the operator  $A$ . There are exactly  $k$  such hyperplanes. In the above coordinates the  $i$ -th of them is defined by the equation  $x_i = 0$ .

The complement to the union of all invariant hyperplanes in the cone  $\pi_+$  consists of  $2^k$  arcwise connected components. Consider one of them.

**Definition 4.2.** The convex hull of all  $\pi$ -integer points except the origin contained in the given arcwise connected component is called a *factor-sail* of the operator  $A$ . The set of all factor-sails is said to be the *factor-continued fraction* for the operator  $A$ .

The union of all orbits  $T_A(*)$  in  $\mathbb{R}^n$  represented by the points in the factor-sail is called the *sail* of the operator  $A$ . The set of all sails is said to be the *continued fraction* for the operator  $A$  (in the sense of Klein-Voronoi).

The intersection of the factor-sail with a hyperplane in  $\pi$  is said to be an  $m$ -dimensional face of the factor-sail if it is homeomorphic to the  $m$ -dimensional disc.

The union of all orbits in  $\mathbb{R}^n$  represented by points in some face of the factor-sail is called the *orbit-face* of the operator  $A$ .

Integer points of the sail are said to be *vertices* of this sail.

**4.2. Algebraic continued fractions.** Consider now an operator  $A$  in the group  $SL(n, \mathbb{Z})$  with irreducible characteristic polynomial. Suppose that it has  $k$  real roots  $r_1, \dots, r_k$  and  $2l$  complex conjugate roots:  $c_1, \bar{c}_1, \dots, c_l, \bar{c}_l$ , where  $k + 2l = n$ . In the simplest possible cases  $k+l = 1$  any factor-sail of  $A$  is a point. If  $k+l > 1$ , than any factor-sail of  $A$  is an infinite polyhedral surface homeomorphic to  $\mathbb{R}^{k+l-1}$ .

The Dirichlet group  $\Xi(A)$  defined in Subsection 3.1 takes any sail of  $A$  to itself. By Dirichlet unit theorem the group  $\Xi(A)$  is homomorphic to  $\mathbb{Z}^{k+l-1}$  and its action on any sail is free. The quotient of a sail by the action of  $\Xi(A)$  is homeomorphic to the  $(n-1)$ -dimensional torus. By a *fundamental domain* of the sail we mean a collection of open orbit-faces such that for any  $\Xi(A)$ -orbit of orbit-faces of the sail there exists a unique representative in the collection.

**Example 4.3.** Let us study an operator  $A$  with a Frobenius matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

This operator has one real and two complex conjugate eigenvalues. Therefore, the cone  $\pi_+$  for  $A$  is a two-dimensional half-plane. In Figure 2a the halfplane  $\pi_+$  is colored in light gray and the invariant plane corresponding to the couple of complex eigenvectors is in

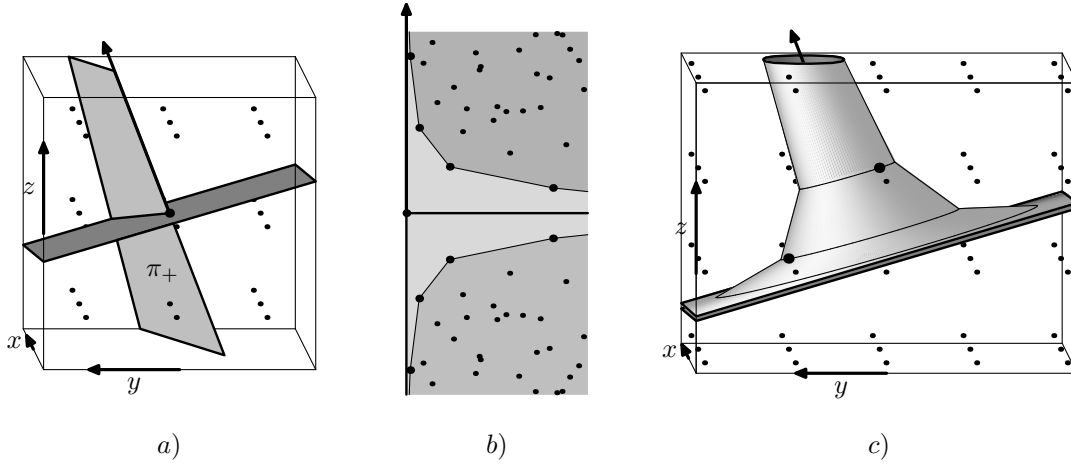


FIGURE 2. A tree-dimensional example: a) the cone  $\pi_+$  and the eigenplane; b) the continued factor-fraction; c) the continued fraction.

dark gray. The vector shown in Figure 2a with endpoint at the origin is an eigenvector of  $A$ .

In Figure 2b we show the cone  $\pi_+$ . The invariant plane separates  $\pi_+$  onto two parts. The dots on  $\pi_+$  are the  $\pi$ -integer points. The boundaries of the convex hulls in each part of  $\pi_+$  are two factor-sails. Actually, the sail corresponding to one factor is taken to the sail corresponding to the other by the operator  $-E$ , where  $E$  is an identity operator of  $\mathbb{R}^3$ .

Finally, in Figure 2c we show one of the sails. Tree orbit-vertices shown in the figure corresponds to the vectors  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(0, 0, 1)$ . You can see the large dark points  $(0, 1, 0)$  and  $(0, 0, 1)$  lying on the corresponding orbit-vertices.

The Dirichlet group in our example is homeomorphic to  $\mathbb{Z}$  and it is generated by  $A$ . The operator  $A$  takes the point  $(1, 0, 0)$  and its orbit-vertex to the point  $(0, 1, 0)$  and the corresponding orbit-vertex. Therefore, a fundamental domain of the operator  $A$  contains one orbit-vertex and one vertex edge. For instance, we can choose the orbit-vertex corresponding to the point  $(1, 0, 0)$  and the orbit-edge corresponding to the "tube" connecting orbit-vectors for the points  $(1, 0, 0)$  and  $(0, 1, 0)$ .

**4.3. Fundamental domains of sails and reduced Hessenberg matrices.** Let us show how to use Klein-Voronoi continued fractions to find minima of MD-characteristics.

**Theorem 4.4.** *For any sail of the multidimensional continued fraction in the sense of Klein-Voronoi of the operator  $A$  we choose one of its fundamental domains. The union of all these fundamental domains contains vertices at which the MD-characteristic  $\Delta(A)(\lambda_1, \dots, \lambda_n)$  attains the minimal values over the integer points of  $\mathbb{R}^n$  except the origin.*

*Remark 4.5.* Any reduced Hessenberg matrix for the operator  $A$  can be constructed starting from some vertex in a fundamental domain of the multidimensional continued fraction

in the sense of Klein-Voronoi. One should construct the Klein-Voronoi continued fraction for the operator  $A$ . Then one takes all vertices of the chosen fundamental domains for all the sails for  $A$ , and finds a vertex  $v$  with minimal value of the MD-characteristic (note that the set of such points is finite). By Corollary 2.3 we reconstruct a basis  $\{e_i\}$  where  $e_1 = v$  in which the operator  $A$  has a perfect Hessenberg matrix  $M$ . By Proposition 3.4 the Hessenberg complexity of any Hessenberg matrix  $M'$  in basis  $\{e'_i\}$  equals to MD-characteristic  $\Delta(A|e'_1)$ . By Theorem 4.4 the vector  $v$  is one of the absolute minima of the MD-characteristic over the integer points of  $\mathbb{R}^n$  except the origin. Therefore, the perfect Hessenberg matrix  $M$  is reduced.

*Proof.* Let  $A$  be an  $SL(n, \mathbb{Z})$ -operator with irreducible characteristic polynomial. By Theorem 3.5 there exists a nonzero constant  $\alpha$  such that MD-characteristic at any point in the system of coordinates  $OX_1X_2 \dots X_kY_1Z_1Y_2Z_2 \dots Y_lZ_l$  is

$$F(x_1, \dots, x_k, y_1, z_1, \dots, y_l, z_l) = \alpha \left| \prod_{i=1}^k x_i \prod_{i=1}^l (y_i^2 + z_i^2) \right|$$

for some positive  $\alpha$ . Suppose that the minimal absolute value of  $F$  on the set of integer points except the origin equals  $m_0$ .

Choose the coordinates  $OX_1 \dots X_kY_1Y_2, \dots, Y_l$  in the cone  $\pi_+$ . Consider a projection of  $\mathbb{R}^n$  to the cone along the  $T_A(v)$  orbits. Since we project along the  $T_A(v)$  orbits on which the MD-characteristic is constant, the projection of the MD-characteristic is well-defined. In the chosen coordinates of  $\pi_+$  it is written as follows:

$$\alpha \left| \prod_{j=1}^k x_j \prod_{j=1}^l y_j^2 \right|.$$

The obtained function is convex in any orthant of the cone  $\pi_+$ . Let the minimal value of the MD-characteristic equals  $m$ . Then for any  $\pi$ -integer point  $v$  except the origin the value of the function is not less than  $m$ . Therefore, the  $m$ -th level of this function intersects the convex hull of all  $\pi$ -integer points of some open orthant in its vertices. By definition, these vertices are the vertices of some factor-sail, and the corresponding integer points are the integer points of the sail. Since the value of the MD-characteristic is invariant under the Dirichlet group action, any fundamental domain of the corresponding sail of the continued fraction contains vertices at which the MD-characteristic attains the minimal value over the integer points of  $\mathbb{R}^n$  except the origin.  $\square$

## 5. THREE-DIMENSIONAL NRS-MATRICES

In this section we study algebraic  $SL(3, \mathbb{Z})$ -matrices with irreducible characteristic polynomials having a real and two complex conjugate roots. We start in Subsection 5.1 with formulation of the original statements in two-dimensional case. In Subsection 5.2 we describe some properties of the subset of NRS-matrices in the family of all algebraic matrices with given Hessenberg type. Further in Subsection 5.3 we show that in any ray consisting of Hessenberg NRS-matrices only finitely many matrices are not reduced. In

Subsection 5.4 we illustrate the obtained results in a few examples. Finally, in Subsection 5.5 we give an example of an operator with two distinct reduced perfect Hessenberg matrices and show that Hessenberg complexity together with characteristic polynomial do not distinguish all the conjugacy classes.

**5.1. A few words about two-dimensional case.** First, we recall a situation in two-dimensional case.

**5.1.1. Classification of conjugacy classes in  $SL(2, \mathbb{Z})$ .** The classification of  $SL(2, \mathbb{Z})$ -operators with complex spectra is finite. Any such operator is integer conjugate to one of the following three operators (that are not conjugate to each other):

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

In the case of operators with multiple eigen-values the complete set of non-conjugate representatives is

$$\left\{ \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \mid k \in \mathbb{Z} \right\}.$$

Finally let us describe the operators with real spectra without multiple eigenvalues.

**Definition 5.1.** Consider a broken line  $\dots V_{k-1}V_kV_{k+1}\dots$ . The *Characteristic sequence* of this broken line is the following two-side infinite sequence of integers

$$\dots, \text{Il}(V_{k-1}V_k), \text{Ia}(V_{k-1}V_kV_{k+1}), \text{Il}(V_kV_{k+1}), \dots$$

where  $\text{Il}(PQ)$  is the integer length of  $PQ$ , and  $\text{Ia}(PQR)$  is the *integer angle* of  $PQR$  (i.e. the index of the lattice generated by all integer points of the vectors  $PQ$  and  $QR$  in the whole lattice).

The Klein-Voronoi continued fraction of an operator  $A$  with real spectra is a collection of four broken lines (sails). It turns out that all four characteristic sequences for them has the same period.

**Theorem 5.2.** Consider an  $SL(2, \mathbb{Z})$ -operator  $H_{(a,c)}^{(b,d)}(m)$ .

i). Let  $a = 0$ ,  $b = 1$ ,  $d = 1$ . If  $m > 2$  then the operator  $A$  has a real spectrum. The corresponding period is

$$(1, m - 1).$$

ii). Let  $b > a \geq 1$ ,  $0 < d \leq b$ ,  $m \geq 1$ . Let the odd ordinary continued fraction for  $b/a$  equal

$$[a_0 : a_1; \dots; a_{2k}].$$

Then the operator  $A$  has a real spectrum. The corresponding period is

$$(a_0, a_1, \dots, a_{2k}, m).$$

□



For the proofs of this theorem we refer to [13].

Notice that for any couple of relatively prime integers  $(a, b)$  where  $b > a \geq 0$  there exists a couple of integers  $(c, d)$ , satisfying  $0 < d \leq b$  and  $ad - bc = 1$ .

For negative values of  $m$  in the case  $a = 0$  the periods are  $(1, |m| - 3)$ . In the case  $a > 0$  the periods equal

$$(a'_{2k}, \dots, a'_1, a'_0, |m| - 2),$$

where  $[a'_0 : a'_1; \dots; a'_{2t}]$  — is the odd ordinary continued fraction for  $b/(b - a)$ .

*Remark 5.3.* The described periods form the complete invariant of operators. Here all cyclically shifted periods are considered to be the same (e.g.  $(a_1 a_2 a_3) = (a_2 a_3 a_1) = (a_3 a_1 a_2)$ ), while the multiple periods not (e.g.  $(a_1 a_2) \neq (a_1 a_2 a_1 a_2)$ ).

For further information about the two-dimensional case we refer the reader, for instance, to the works [20], [21], and [13].

**5.1.2. Family of matrices with fixed Hessenberg type.** In the next subsections we generalize the following statement.

**Statement 5.4.** *Consider the family of matrices  $H_{\Omega}^{v_0}(m)$  with parameter  $m$ . Then almost all matrices in this family are reduced and has two real eigenvalues.*  $\square$

The proof can be easily deduced from Theorem 6 of [13].

**Example 5.5.** Consider the matrices of the Hessenberg type  $\langle 2, 5 \rangle$ . Let

$$H_{\langle 2, 5 \rangle}^{(1, 3)}(m) = \begin{pmatrix} 2 & 1 + 2m \\ 5 & 3 + 5m \end{pmatrix}.$$

The Hessenberg complexity of all these matrices is 5. There are no matrices with reducible characteristic polynomial in the family. The matrix with  $m = -1$  has non-real spectrum, and the matrices with  $m = -3, -2, 0, 1$  are nonreduced. All the rest matrices of that type are reduced, have two real eigenvalues, and with irreducible characteristic polynomial.

The corresponding periods for the case of  $m \geq 2$  are:  $(2, 1, 1, m)$ ; for the case  $m \leq -4$  they are  $(1, 2, 1, -m - 2)$ .

## 5.2. Asymptotic structure of the subset of perfect Hessenberg NRS-matrices.

Note that an  $SL(3, \mathbb{Z})$ -matrix has a reducible characteristic polynomial iff one of its eigenvalues equals to  $\pm 1$ .

In Theorem 2.1 we have shown that any  $SL(3, \mathbb{Z})$ -matrix is integer conjugate to one of the reduced Hessenberg matrices of the form:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ 0 & a_{3,2} & a_{3,3} \end{pmatrix},$$

where  $0 \leq a_{1,2} < a_{2,1}$ ,  $0 \leq a_{1,2} < a_{3,2}$ , and  $0 \leq a_{2,2} < a_{3,2}$ .

Consider the family  $H(\Omega)$  for a Hessenberg type  $\Omega = \langle a_{1,1}, a_{2,1} | a_{1,2}, a_{2,2}, a_{3,2} \rangle$ . Denote the subset of all NRS-matrices in  $H(\Omega)$  by  $NRS(\Omega)$ .

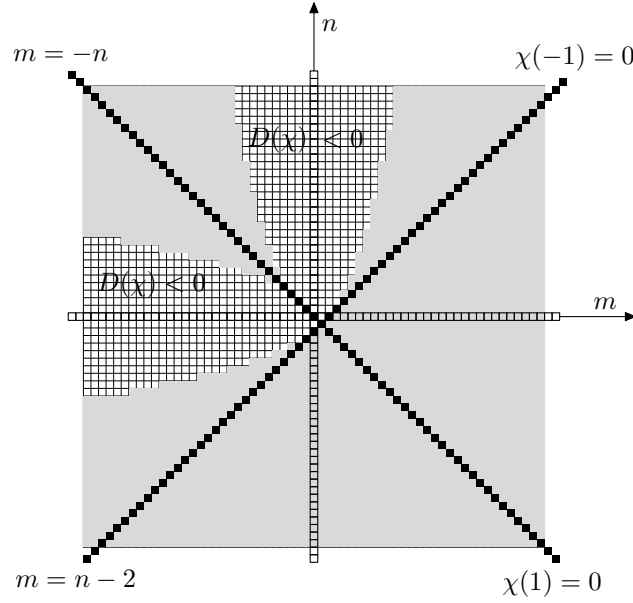


FIGURE 3. The family of matrices of Hessenberg type  $\langle 0, 1|0, 0, 1 \rangle$ .

For the given Hessenberg type  $\Omega$  we choose integers  $a_{1,3}$ ,  $a_{2,3}$ , and  $a_{3,3}$  such that the determinant of the matrix  $M_0 = (a_{i,j})$  is a unit. As we have shown in Corollary 2.7 the set  $H(\Omega)$  is a family

$$\left\{ \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,1}m + a_{1,2}n + a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,1}m + a_{2,2}n + a_{2,3} \\ 0 & a_{3,2} & a_{3,2}n + a_{3,3} \end{pmatrix} \middle| m \in \mathbb{Z}, n \in \mathbb{Z} \right\}.$$

By  $H_\Omega^{v_0}(m_0, n_0)$  we denote the matrix with integer parameters  $m = m_0$  and  $n = n_0$  in the family  $H(\Omega)$ . Later in the proofs we use also matrices  $H_\Omega(m_0, n_0)$  with real parameters, i.e., the matrices in the real affine two-dimensional plane spanned by  $H(\Omega)$  in the real affine space of all matrices. We denote by  $OMN$  the coordinate system corresponding to the parameters  $(m, n)$ . The origin  $O$  here corresponds to the matrix  $H_\Omega(0, 0)$ .

Denote the discriminant of the characteristic polynomial of  $H_\Omega(m, n)$  by  $\mathcal{D}_\Omega(m, n)$ . So the set  $NRS(\Omega)$  coincides with the set of integer solutions of the inequality

$$\mathcal{D}_\Omega(m, n) < 0$$

in variables  $m$  and  $n$ .

**Example 5.6.** We show in Figure 3 the subset of NRS-matrices  $NRS(\Omega)$  for the Hessenberg type  $\Omega = \langle 0, 1|0, 0, 1 \rangle$ . For this example we choose  $(a_{1,3}, a_{2,3}, a_{3,3}) = (0, 0, 1)$ .

In both examples shown in Figure 3 and in Figure 1 on page 9 we see that the set  $NRS(\Omega)$  "looks like" the set of integer points in the union of the convex hulls of two parabolas.

Let us formulate a precise statement. Suppose that the characteristic polynomial of the matrix  $H_\Omega^{v_0}(0, 0) = (a_{i,j})$  in the variable  $t$  equals

$$-t^3 + b_1 t^2 - b_2 t + b_3.$$

In the case of  $SL(3, \mathbb{Z})$  we have  $b_3 = 1$ , nevertheless we continue to write  $b_3$  for possible use for matrices with distinct determinants.

For a family  $H_\Omega(m, n)$  we define the following two quadratic functions

$$\begin{aligned} p_{1,\Omega}(m, n) &= m - \alpha_1 n^2 - \beta_1 n - \gamma_1; \\ p_{2,\Omega}(m, n) &= \frac{n}{a_{2,1}} - \alpha_2 \left( \frac{a_{2,1}m - a_{1,1}n}{a_{2,1}} \right)^2 - \beta_2 \left( \frac{a_{2,1}m - a_{1,1}n}{a_{2,1}} \right) - \gamma_2, \end{aligned}$$

where

$$\begin{cases} \alpha_1 = -\frac{a_{3,2}}{4a_{2,1}} \\ \beta_1 = \frac{a_{1,1} - a_{2,2} - a_{3,3}}{2a_{2,1}} \\ \gamma_1 = \frac{4b_2 - b_1^2}{4a_{2,1}a_{3,2}} \end{cases} ; \quad \begin{cases} \alpha_2 = \frac{a_{3,2}a_{2,1}}{4b_3} \\ \beta_2 = -\frac{b_2}{2b_3} \\ \gamma_2 = \frac{b_2^2 - 4b_1b_3}{4a_{2,1}a_{3,2}b_3} \end{cases}.$$

Denote by  $B_R(O)$  the interior of the circle of radius  $R$  centered at the origin  $(0, 0)$  in the real plane  $OMN$  of the family  $H_\Omega(m, n)$ . We denote also

$$\Lambda_\varepsilon = \{(m, n) | (p_{1,\Omega}(m, n) - \varepsilon)(p_{2,\Omega}(m, n) - \varepsilon) < 0\}.$$

**Theorem 5.7.** *For any positive  $\varepsilon$  there exists  $R > 0$  such that in the complement to  $B_R(O)$  the following inclusions hold*

$$\Lambda_\varepsilon \subset NRS(\Omega) \subset \Lambda_{-\varepsilon}.$$

Before to start the proof we make the following remark. The set  $NRS(\Omega)$  is defined by the inequality

$$\mathcal{D}_\Omega^{v_0}(m, n) < 0.$$

In the left part of the inequality there is a polynomial of degree 4 in variables  $m$  and  $n$ . Note that the product  $16a_{2,1}^2 a_{3,2}^2 b_3 (p_{1,\Omega}(m, n) p_{2,\Omega}(m, n))$  is a good approximation to  $\mathcal{D}_\Omega^{v_0}(m, n)$  at infinity: the polynomial

$$\mathcal{D}_\Omega(m, n)^{v_0} - 16a_{2,1}^2 a_{3,2}^2 b_3 (p_{1,\Omega}(m, n) p_{2,\Omega}(m, n))$$

is a polynomial of degree 2 in variables  $m$  and  $n$ .

**Lemma 5.8.** *The curve  $\mathcal{D}_{\langle 0,1|0,0,1 \rangle}^{(1,0,0)}(m, n) = 0$  is contained in the domain defined by the inequalities:*

$$\begin{cases} (m^2 - 4n + 3)(n^2 + 4m + 3) \geq 0 \\ (m^2 - 4n - 3)(n^2 + 4m - 3) - 72 \leq 0 \end{cases}$$

*Remark.* Lemma 5.8 implies that the curve  $\mathcal{D}_{\langle 0,1|0,0,1 \rangle}^{(1,0,0)}(m, n) = 0$  is contained in some tubular neighborhood of the curve

$$(m^2 - 4n)(n^2 + 4m) = 0.$$

*Proof.* Note that

$$\mathcal{D}_{\langle 0,1|0,0,1 \rangle}^{(1,0,0)}(m, n) = (m^2 - 4n)(n^2 + 4m) - 2mn - 27.$$

Thus, we have

$$\mathcal{D}_{\langle 0,1|0,0,1 \rangle}^{(1,0,0)}(m, n) - (m^2 - 4n + 3)(n^2 + 4m + 3) = -2(n - 3)^2 - 2(m + 3)^2 - (n + m)^2 \leq 0,$$

and

$$\mathcal{D}_{\langle 0,1|0,0,1 \rangle}^{(1,0,0)}(m, n) - (m^2 - 4n - 3)(n^2 + 4m - 3) + 72 = 2(n - 3)^2 + 2(m + 3)^2 + (n - m)^2 \geq 0.$$

Therefore, the curve  $\mathcal{D}_{\langle 0,1|0,0,1 \rangle}^{(1,0,0)}(m, n) = 0$  is contained in the domain defined in the lemma.  $\square$

**Lemma 5.9.** *For any  $\Omega = \langle a_{11}, a_{21} | a_{12}, a_{22}, a_{32} \rangle$  there exists a (not necessarily integer) affine transformation of the plane  $OMN$  taking the curve  $\mathcal{D}_{\Omega}^{v_0}(m, n) = 0$  to the curve  $\mathcal{D}_{\langle 0,1|0,0,1 \rangle}^{(1,0,0)}(m, n) = 0$ .*

*Proof.* Let  $H_{\Omega}(0, 0)^{v_0} = (a_{i,j})$ . Note that a matrix  $H_{\Omega}^{v_0}(m, n)$  is rational conjugate to the matrix

$$H_{\langle 0,1|0,0,1 \rangle}^{(1,0,0)}(a_{23}a_{32} - a_{11}a_{33} + a_{12}a_{21} - a_{22}a_{33} - a_{11}a_{22} + a_{21}a_{32}m - a_{11}a_{32}n, a_{11} + a_{22} + a_{33} + a_{32}n)$$

by the matrix

$$X_{\Omega}^{v_0} = \begin{pmatrix} 1 & a_{1,1} & a_{1,1}^2 + a_{1,2}a_{2,1} \\ 0 & a_{2,1} & a_{1,1}a_{2,1} + a_{2,1}a_{2,2} \\ 0 & 0 & a_{2,1}a_{3,2} \end{pmatrix}.$$

Since both matrices

$$H_{\Omega}^{v_0}(m, n) \quad \text{and} \quad (X_{\Omega}^{v_0})^{-1} (H_{\Omega}(m, n)) X_{\Omega}^{v_0}$$

have the equivalent characteristic polynomials, their discriminants coincide. Therefore, the curve  $\mathcal{D}_{\Omega}^{v_0} = 0$  is mapped to the curve  $\mathcal{D}_{\langle 0,1|0,0,1 \rangle}^{(1,0,0)}(m, n) = 0$  bijectively.

In  $OMN$  coordinates this map corresponds to the following affine transformation

$$\begin{pmatrix} m \\ n \end{pmatrix} \mapsto \begin{pmatrix} a_{21}a_{32}m - a_{11}a_{32}n \\ a_{32}n \end{pmatrix} + \begin{pmatrix} a_{23}a_{32} - a_{11}a_{33} + a_{12}a_{21} - a_{22}a_{33} - a_{11}a_{22} \\ a_{11} + a_{22} + a_{33} \end{pmatrix}.$$

This completes the proof of the lemma.  $\square$

*Proof of Theorem 5.7.* Consider a family of matrices  $H_{\Omega}^{v_0}(-p_{1,\Omega}(0, t) + \varepsilon, t)$  with real parameter  $t$ . Direct calculations show that for  $\varepsilon \neq 0$  the discriminant of the matrices for this family is a polynomial of the forth degree in variable  $t$ , and

$$\mathcal{D}_{\Omega}^{v_0}(-p_{1,\Omega}(0, t) + \varepsilon, t) = \frac{1}{4}a_{2,1}a_{3,2}^5\varepsilon t^4 + O(t^3).$$

Therefore, there exists a neighborhood of infinity with respect to the variable  $t$  such that the function  $\mathcal{D}_{\Omega}^{v_0}(-p_{1,\Omega}(0, t) + \varepsilon, t)$  is positive for positive  $\varepsilon$  in the neighborhood, and negative for negative  $\varepsilon$ .

Hence for a given  $\varepsilon$  there exists a sufficiently large  $N_1 = N_1(\varepsilon)$  such that for any  $t > N_1$  there exists a solution of the equation  $\mathcal{D}_\Omega^{v_0}(m, n) = 0$  at the segment with endpoints

$$(-p_{1,\Omega}(0, t) + \varepsilon, t) \quad \text{and} \quad (-p_{1,\Omega}(0, t) - \varepsilon, t)$$

of the plane  $OMN$ .

Now we examine the family in variable  $t$  for the second parabola:

$$H_\Omega^{v_0} \left( t - a_{1,1}p_{2,\Omega}(t, 0) - \frac{a_{1,1}}{\sqrt{a_{1,1}^2 + a_{2,1}^2}}\varepsilon, -a_{2,1}p_{2,\Omega}(t, 0) - \frac{a_{2,1}}{\sqrt{a_{1,1}^2 + a_{2,1}^2}}\varepsilon \right).$$

By the same reasons, for a given  $\varepsilon$  there exists a sufficiently large  $N_2 = N_2(\varepsilon)$  such that for any  $t > N_2$  there exists a solution of the equation  $\mathcal{D}_\Omega(m, n)^{v_0} = 0$  at the segment with endpoints

$$\begin{aligned} & \left( t - a_{1,1}p_{2,\Omega}(t, 0) - \frac{a_{1,1}}{\sqrt{a_{1,1}^2 + a_{2,1}^2}}\varepsilon, -a_{2,1}p_{2,\Omega}(t, 0) - \frac{a_{2,1}}{\sqrt{a_{1,1}^2 + a_{2,1}^2}}\varepsilon \right) \quad \text{and} \\ & \left( t - a_{1,1}p_{2,\Omega}(t, 0) + \frac{a_{1,1}}{\sqrt{a_{1,1}^2 + a_{2,1}^2}}\varepsilon, -a_{2,1}p_{2,\Omega}(t, 0) + \frac{a_{2,1}}{\sqrt{a_{1,1}^2 + a_{2,1}^2}}\varepsilon \right) \end{aligned}$$

of the plane  $OMN$ .

We have shown that for any of the four branches of the two parabolas defined by  $p_{1,\Omega}$  and  $p_{2,\Omega}$  there exists (at least) one branch of  $\mathcal{D}_\Omega(m, n)^{v_0} = 0$  contained in the  $\varepsilon$ -tube of the chosen parabolic branch if we are far enough from the origin.

In Lemma 5.8 we have obtained that  $\mathcal{D}_{\langle 0,1|0,0,1 \rangle}^{(1,0,0)}(m, n) = 0$  is contained in some tubular neighborhood of  $p_{1,\langle 0,1|0,0,1 \rangle}(m, n)p_{2,\langle 0,1|0,0,1 \rangle}(m, n) = 0$ . Then by Lemma 5.9 we have that the curve  $\mathcal{D}_\Omega^{v_0}(m, n) = 0$  is contained in some tubular neighborhood of the curve  $p_{1,\Omega}(m, n)p_{2,\Omega}(m, n) = 0$  outside some ball centered at the origin. Finally, by Viet Theorem, the intersection of the curve  $\mathcal{D}_\Omega^{v_0}(m, n) = 0$  with each of the parallel lines

$$\ell_t : \frac{a_{1,1} + a_{2,1}}{a_{2,1}}n - m = t$$

contains at most four points. Therefore, there exists sufficiently large  $T$  such that for any  $t \geq T$  the intersection of the curve  $\mathcal{D}_\Omega^{v_0}(m, n) = 0$  and  $\ell_t$  contains exactly four points corresponding to the branches of the parabolas  $p_{1,\Omega}(m, n) = 0$  and  $p_{2,\Omega}(m, n) = 0$  lying in  $\Lambda_{-\varepsilon} \setminus \Lambda_\varepsilon$ .

Hence, there exists  $R = R(\varepsilon, N_1, N_2, T)$  such that in the complement to the ball  $B_R(O)$  we have

$$\Lambda_\varepsilon \subset NRS(\Omega) \subset \Lambda_{-\varepsilon}.$$

The proof of Theorem 5.7 is completed.  $\square$

**5.3. Asymptotic behaviour of Klein-Voronoi continued fractions for the NRS-case.** In the previous subsection we have shown that for any Hessenberg type  $\Omega = \langle a_{1,1}, a_{2,1} | a_{1,2}, a_{2,2}, a_{3,2} \rangle$  the set  $NRS(\Omega)$  almost coincides with the union of the convex hulls of two parabolas. We say that the set  $NRS(\Omega)$  has "two asymptotic directions" that correspond to the symmetry lines of the parabolas. These directions are defined by the vectors  $(-1, 0)$  and  $(a_{1,1}, a_{2,1})$ . To be precise, an integer ray in  $OMN$  is said to be

an *NRS-ray* if all its integer points correspond to reduced Hessenberg NRS-matrices. A direction is said to be *asymptotic* for the set  $NRS(\Omega)$  if there exists an NRS-ray with this direction.

Theorem 5.7 implies the following proposition.

**Proposition 5.10.** *There are exactly two asymptotic directions for the set  $NRS(\Omega)$ , they are defined by the vectors  $(-1, 0)$  and  $(a_{1,1}, a_{2,1})$ .*  $\square$

We will use the following notation. Fix a basis  $\{e_1, e_2, e_3\}$  in  $\mathbb{R}^3$ , consider a Hessenberg type  $\Omega = \langle a_{1,1}, a_{2,1} | a_{1,2}, a_{2,2}, a_{3,2} \rangle$  and choose an integer appropriate vector  $v_0$ . For any couple of real numbers  $(m, n)$  and nonnegative  $t$  we denote the operators  $R_{m,n}^{1,\Omega}(t)$  and  $R_{m,n}^{2,\Omega}(t)$  with matrices:

$$H_{\Omega}^{v_0}(m - t, n) \quad \text{and} \quad H_{\Omega}^{v_0}(m + a_{1,1}t, n + a_{2,1}t)$$

respectively. Denote by  $R_{m,n}^{1,\Omega}$  and  $R_{m,n}^{2,\Omega}$  the two corresponding families of operators with nonnegative parameter  $t$ . The directions of these rays are asymptotic:  $(-1, 0)$  and  $(a_{1,1}, a_{2,1})$  respectively.

Now we are ready to formulate and to prove the following theorem.

**Theorem 5.11. On asymptotics for NRS-rays.** *Any NRS-ray for the set  $NRS(\Omega)$  contains only finitely many nonreduced matrices. Any such ray contains only finitely many reduced matrices integer conjugate to some other reduced matrices.*

**Example 5.12.** Any NRS-ray for the Hessenberg type  $\langle 0, 1 | 0, 0, 1 \rangle$  contains only reduced perfect matrices. Experiments show that any NRS-ray for  $\langle 0, 1 | 1, 0, 2 \rangle$  contains at most one nonreduced matrix.

*Proof.* First, we prove the theorem for NRS-rays with asymptotic direction  $(-1, 0)$ .

Let us begin with the case of real matrices of Hessenberg type  $\Omega_0 = \langle 0, 1 | 0, 0, 1 \rangle$ . Such matrices form a family  $H(\Omega_0)$  with real parameters  $m$  and  $n$  as before:

$$H_{\langle 0, 1 | 0, 0, 1 \rangle}^{(1, 0, 0)}(m, n) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & m \\ 0 & 1 & n \end{pmatrix}.$$

**Lemma 5.13.** *Let  $R_{m,n}^{1,\Omega}$  be a family of operators having NRS-matrices with nonnegative integer parameter  $t$ . Then for any  $\varepsilon > 0$  there exists  $C > 0$  such that for any  $t > C$  the convex hull of the union of two orbit-vertices*

$$T_{R_{m,n}^{1,\Omega_0}(t)}(1, 0, 0) \quad \text{and} \quad T_{R_{m,n}^{1,\Omega_0}(t)}(0, 1, 0)$$

*is contained in the  $\varepsilon$ -tubular neighborhood of the convex hull of three points  $(1, 0, 0)$ ,  $(0, 1, 0)$ ,  $(0, -1, 0)$ .*

*Proof.* Let us find the asymptotics of eigenvectors and eigenplanes for operators  $R_{m,n}^{1,\Omega_0}(t)$  while  $t$  tends to  $+\infty$ . Denote any real eigenvector of  $R_{m,n}^{1,\Omega_0}(t)$  by  $e(t)$ . We have

$$e(t) = \mu((1, 0, 0) + O(t^{-1}))$$

for some nonzero real  $\mu$ .

Consider the unique invariant real plane of the operator  $R_{m,n}^{1,\Omega_0}(t)$  (it corresponds to the couple of complex conjugate eigenvalues). Note that this plane is a union of all closed orbits of  $R_{m,n}^{1,\Omega_0}(t)$ . Any such orbit is an ellipse with axes  $\lambda g_{\max}(t)$  and  $\lambda g_{\min}(t)$  for some positive real number  $\lambda$ , where

$$\begin{aligned} g_{\max}(t) &= (0, t, 0) + O(1), \\ g_{\min}(t) &= (0, 0, t^{1/2}) + O(t^{-1/2}). \end{aligned}$$

Actually, the vectors  $g_{\max}(t) \pm I g_{\min}(t)$  are two complex eigenvectors of  $R_{m,n}^{1,\Omega_0}(t)$ . For the ratio of the lengths of maximal and minimal axes of any orbit we have the following asymptotic estimate:

$$\frac{\lambda |g_{\max}(t)|}{\lambda |g_{\min}(t)|} = |t|^{1/2} + O(|t|^{-1/2}).$$

Since

$$(1, 0, 0) - \frac{1}{\mu} e(t) = O(|t|^{-1}),$$

the minimal axis of the orbit-vertex  $T_{R_{m,n}^{1,\Omega_0}(t)}(1, 0, 0)$  is asymptotically not greater than  $O(t^{-1})$ . Therefore, the length of the maximal axis is asymptotically not greater than some function  $O(|t|^{-1/2})$ . Hence, the orbit of the point  $(1, 0, 0)$  is contained in the  $(C_1 |t|^{-1/2})$ -ball of the point  $(1, 0, 0)$ , where  $C_1$  is a constant that does not depend on  $t$ .

We have

$$(0, 1, 0) - \frac{1}{t} g_{\max}(t) = O(|t|^{-1}).$$

Therefore, the length of the maximal axis of the orbit-vertex  $T_{R_{m,n}^{1,\Omega_0}(t)}(1, 0, 0)$  is asymptotically not greater than some  $1 + O(t^{-1/2})$ . Hence, the length of the minimal axis is asymptotically not greater than some  $O(|t|^{-1/2})$ . This implies that the orbit of the point  $(0, 1, 0)$  is contained in the  $(C_2 |t|^{-1/2})$ -tubular neighborhood of the segment with vertices  $(0, 1, 0)$  and  $(0, -1, 0)$ , where  $C_2$  is a constant that does not depend on  $t$ .

Therefore, the convex hull of the union of two orbit-vertices

$$T_{R_{m,n}^{1,\Omega_0}(t)}(1, 0, 0) \quad \text{and} \quad T_{R_{m,n}^{1,\Omega_0}(t)}(0, 1, 0)$$

is contained in the  $C$ -tubular neighborhood of the triangle with vertices  $(1, 0, 0)$ ,  $(0, 1, 0)$ ,  $(0, -1, 0)$ , where  $C = |t|^{-1/2} \max(C_1, C_2)$ . This concludes the proof of the lemma.  $\square$

Let us now formulate a similar statement for the general case of Hessenberg operators.

**Corollary 5.14.** *Let  $R_{m,n}^{1,\Omega}$  be a family of operators having NRS-matrices with integer parameter  $t \geq 0$ . Then for any  $\varepsilon > 0$  there exists  $C > 0$  such that for any  $t > C$  the convex hull of the union of two orbit-vertices*

$$T_{R_{m,n}^{1,\Omega_0}(t)}(1, 0, 0) \quad \text{and} \quad T_{R_{m,n}^{1,\Omega_0}(t)}(a_{1,1}, a_{2,1}, 0)$$

*is contained in the  $\varepsilon$ -tubular neighborhood of the convex hull of three points  $(1, 0, 0)$ ,  $(a_{1,1}, a_{2,1}, 0)$ ,  $(-a_{1,1}, -a_{2,1}, 0)$ .*

*Proof.* Denote  $\Omega = \langle a_{1,1}, a_{2,1} | a_{1,2}, a_{2,2}, a_{3,2} \rangle$  and choose

$$X = \begin{pmatrix} a_{2,1}a_{3,2} & -a_{3,2}a_{1,1} & a_{1,1}a_{2,2} - a_{2,1}a_{1,2} \\ 0 & a_{3,2} & -a_{1,1} - a_{2,2} \\ 0 & 0 & 1 \end{pmatrix}.$$

Direct calculation shows that

$$XH_\Omega(m-t, n)X^{-1} = H_{\langle 0,1|0,0,1 \rangle}^{(1,0,0)} \left( l_1(m, n) - \frac{t}{a_{2,1}a_{3,2}}, l_2(n_0) \right),$$

where  $l_1$  and  $l_2$  are linear functions with coefficients depending only on  $a_{1,1}$ ,  $a_{2,1}$ ,  $a_{1,2}$ ,  $a_{2,2}$ , and  $a_{3,2}$ .

Therefore, the family  $R_{m,n}^{1,\Omega}$  after the described change of coordinates and a homothety is taken to the family of matrices  $R_{\tilde{m},\tilde{n}}^{1,\Omega_0}(t)$  of the type  $\langle 0,1|0,0,1 \rangle$  for certain  $\tilde{m}$  and  $\tilde{n}$ .

Lemma 5.13 implies the following. For any  $\varepsilon > 0$  there exists a positive constant such that for any  $t$  greater than this constant the convex hull of the union of two orbit-vertices

$$T_{R_{\tilde{m},\tilde{n}}^{1,\Omega_0}(t)}(1,0,0) \quad \text{and} \quad T_{R_{\tilde{m},\tilde{n}}^{1,\Omega_0}(t)}(0,1,0)$$

is contained in the  $\varepsilon$ -tubular neighborhood of the triangle with vertices  $(1,0,0)$ ,  $(0,1,0)$ ,  $(0,-1,0)$ .

Now if we reformulate the last statement for the family of operators in old coordinates, then we get the statement of the corollary.  $\square$

In the algebraic case we have the following statement.

**Proposition 5.15.** *Let  $(m, n)$  be a couple of integers such that the ray  $R_{m,n}^{1,\Omega}$  contains only operators having NRS-matrices for integer parameter  $t \geq 0$ . Then there exists  $C > 0$  such that for any integer  $t > C$  there exists a fundamental domain for a Klein-Voronoi sail of the operator  $R_{m,n}^{1,\Omega}(t)$  such that all (integer) orbit-vertices of this fundamental domain are contained in the set of all integer orbits corresponding to the integer points in the convex hull of three points  $(1,0,0)$ ,  $(a_{1,1}, a_{2,1}, 0)$ , and  $(-a_{1,1}, -a_{2,1}, 0)$ .*

We use the following general lemma on continued fractions.

Take an operator  $A$  having an NRS-matrix in  $SL(3, \mathbb{Z})$  and any integer point  $x$  distinct from the origin. Denote by  $\Gamma_A^0(p)$  the convex hull of the union of two orbits corresponding to the points  $p$  and  $A(p)$ . For any integer  $k$  we denote by  $\Gamma_A^k(p)$  the set  $A^k(\Gamma_A^0(x))$ . Let

$$\Gamma_A(p) = \bigcup_{k \in \mathbb{Z}} \Gamma_A^k(p).$$

**Lemma 5.16.** *Consider  $A \in SL(3, \mathbb{Z})$  with NRS-matrix and let  $p$  be any integer point distinct from the origin. Then one of the Klein-Voronoi sails for  $A$  is contained in the set  $\Gamma_A(p)$ .*

*Proof.* Note that the set  $\Gamma_A(p)$  is a union of orbits. Let us project  $\Gamma_A(p)$  to the halfplane  $\pi_+$ , see in Figure 2 above. The set  $\Gamma_A(p)$  projects to the closure of the complement of the convex hull for the points  $\pi(A^k(p))$  for all integer  $k$  in the angle defined by eigenspaces.



Since all the points  $A^k(p)$  are integer, their convex hull is contained in the convex hull of all points corresponding to integer orbits in the angle. Hence  $\pi(\Gamma_A(p))$  contains the projection of the sail. Therefore, the set  $\Gamma_A(p)$  contains one of the sails.  $\square$

**Corollary 5.17.** *Let  $A$  be an operator in  $SL(3, \mathbb{Z})$  having an NRS-matrix and  $p$  — an integer point distinct from the origin. Then there exists a fundamental domain for one of the Klein-Voronoi sails for an operator  $A$  with all (integer) orbit-vertices contained in the set  $\Gamma_A^0(p)$ .*

*Proof.* Note that  $\Gamma_A^0(p)$  is a fundamental domain of  $\Gamma_A(p)$  for the action of the Dirichlet group  $\Xi(A)$ . Hence,  $\Gamma_A(p)$  contains all orbits of orbit-vertices for the action of  $\Xi(A)$ .  $\square$

*Proof of Proposition 5.15.* We note that the operator  $R_{m,n}^{1,\Omega}(t)$  takes the point  $(1, 0, 0)$  to the point  $(a_{1,1}, a_{2,1}, 0)$ . Therefore, the convex hull of the union of two orbit-vertices

$$T_{R_{m,n}^{1,\Omega}(t)}(1, 0, 0) \quad \text{and} \quad T_{R_{m,n}^{1,\Omega}(t)}(a_{1,1}, a_{2,1}, 0)$$

(we denote it by  $W(t)$ ) coincides with the set  $\Gamma_{R_{m,n}^{1,\Omega}(t)}^0(1, 0, 0)$ .

From Corollary 5.17 it follows that there exists a fundamental domain for a sail with all its orbit-vertices contained in  $W(t)$ . Choose a sufficiently small  $\varepsilon_0$  such that the  $\varepsilon_0$ -tubular neighborhood of the triangle with vertices  $(1, 0, 0)$ ,  $(a_{1,1}, a_{2,1}, 0)$ , and  $(-a_{1,1}, -a_{2,1}, 0)$  does not contain integer points distinct from the points of the triangle. From Corollary 5.14 it follows that for a sufficiently large  $t$  the set  $W(t)$  is contained in the  $\varepsilon_0$ -tubular neighborhood of the triangle. This implies the statement of Proposition 5.15.  $\square$

Now let us study the remaining case of the rays of matrices with asymptotic direction  $(a_{1,1}, a_{2,1})$ . We remind that  $\Omega_0 = \langle 0, 1 | 0, 0, 1 \rangle$ .

**Lemma 5.18.** *Let  $R_{m,n}^{2,\Omega_0}$  be an NRS-ray. Then for any  $\varepsilon > 0$  there exists  $C > 0$  such that for any  $t > C$  the convex hull of the union of two orbit-vertices  $T_{R_{m,n}^{2,\Omega_0}(t)}(1, 0, 0)$  and  $T_{R_{m,n}^{2,\Omega_0}(t)}(0, 1, 0)$  is contained in the  $\varepsilon$ -tubular neighborhood of the convex hull of three points  $(1, 0, 0)$ ,  $(-1, 0, 0)$ ,  $(0, 1, 0)$ .*

*Proof.* First, we note that the continued fractions for the operators  $A$  and  $A^{-1}$  coincide.

Secondly, the following holds:

$$H_{\langle 0,1|0,0,1 \rangle}^{(1,0,0)}(m, n+t) = X H_{\langle 0,1|0,0,1 \rangle}^{(1,0,0)}(-n-t, -m) X^{-1},$$

where

$$X = \begin{pmatrix} 0 & -1 & -n-t \\ -1 & 0 & -m \\ 0 & 0 & -1 \end{pmatrix}.$$

Thus, in the new coordinates we obtain the equivalent statement for the ray  $R_{-n,-m}^{1,\Omega_0}(t)$ . Now Lemma 5.18 follows directly from Lemma 5.13.  $\square$

**Corollary 5.19.** *Let  $\Omega = \langle a_{1,1}, a_{2,1} | a_{1,2}, a_{2,2}, a_{3,2} \rangle$  and  $R_{m,n}^{2,\Omega}(t)$  be a family of operators having NRS-matrices with parameter  $t \geq 0$ . Then for any  $\varepsilon > 0$  there exists  $C > 0$  such that for any  $t > C$  the convex hull of the union of two orbit-vertices*

$$T_{H_{\Omega}^{v_0}(m+a_{1,1}t, n+a_{2,1}t)}(1, 0, 0) \quad \text{and} \quad T_{H_{\Omega}^{v_0}(m+a_{1,1}t, n+a_{2,1}t)}(a_{1,1}, a_{2,1}, 0)$$

*is contained in the  $\varepsilon$ -tubular neighborhood of the triangle with vertices  $(1, 0, 0)$ ,  $(-1, 0, 0)$ , and  $(a_{1,1}, a_{2,1}, 0)$ .  $\square$*

**Proposition 5.20.** *Let  $(m, n)$  be a couple of integers, such that the family  $R_{m,n}^{2,\Omega}(t)$  contains only operators having NRS-matrices. Then there exists  $C > 0$  such that for any  $t > C$  there exists a fundamental domain for a Klein-Voronoi sail of the operator  $R_{m,n}^2(t)$  such that all orbit-vertices of this fundamental domain are contained in the set of all integer orbits corresponding to the integer points in the convex hull of three points  $(1, 0, 0)$ ,  $(-1, 0, 0)$ , and  $(a_{1,1}, a_{2,1}, 0)$ .  $\square$*

*Remark* We omit the proofs of Corollary 5.19 and Proposition 5.20, since they repeat the proofs of Corollary 5.14 and Proposition 5.15.

Now we prove Theorem 5.11.

*Step 1.* Let  $A$  be an operator with Hessenberg matrix  $M$  in  $SL(3, \mathbb{Z})$ . By Proposition 3.4 the Hessenberg complexity of the Hessenberg matrix  $M$  coincides with the MD-characteristic  $\Delta(A|(1, 0, 0))$ . Therefore, the Hessenberg matrix  $M$  is reduced if and only if the MD-characteristic of  $A$  attains the minimal possible absolute value on the integer lattice except the origin exactly at point  $(1, 0, 0)$ .

*Step 2.* By Theorem 4.4 we know that all minima of the set of absolute values for the MD-characteristic of  $A$  are attained at integer points of the Klein-Voronoi sails for  $A$ .

*Step 3.* In the case of a three-dimensional operator  $A$  with two complex-conjugate eigenvalues, the continued fraction of  $A$  contains exactly two sails that are symmetric with respect to the transformation  $-E$ . Therefore, both sails of  $A$  has minima of the MD-characteristic on an integer lattice except the origin. This allows us to consider one sail. By Theorem 4.4 we can restrict the search of the minimal absolute value of the MD-characteristic to one of the fundamental domains of the chosen sail.

*Step 4.1. The case of NRS-rays with asymptotic direction  $(-1, 0)$ .*

By Proposition 5.15 there exists a positive constant such that for any integer  $t$  greater than this constant all integer points of one of the fundamental domains for  $R_{m,n}^{1,\Omega}(t)$  are contained in the convex hull of three points  $(1, 0, 0)$ ,  $(a_{1,1}, a_{2,1}, 0)$ , and  $(-a_{1,1}, -a_{2,1}, 0)$ .

This triangle contains only finitely many integer points, all of them have the last coordinate equal to zero. The value of the MD-characteristic for the points of type  $(x, y, 0)$  equals:

$$(a_{2,1}x - a_{1,1}y)a_{3,2}^2y^2t + C,$$

where the constant  $C$  does not depend on  $t$ , it depends only on  $x, y$ , and the elements of the matrix  $H_{\Omega}(m - t, n)$  for the operator  $R_{m,n}^{1,\Omega}(t)$ .

So for any point  $(x, y, 0)$  the MD-characteristic is linear with respect to the parameter  $t$ , and it increases with growth of  $t$ . The only exceptions are the points of type  $\lambda(1, 0, 0)$

and  $\mu(a_{1,1}, a_{2,1}, 0)$  (for integers  $\lambda$  and  $\mu$ ). The values of MD-characteristic are constant in these points with respect to the parameter  $t$ .

Since there are finitely many integer points in the triangle  $(1, 0, 0)$ ,  $(a_{1,1}, a_{2,1}, 0)$ , and  $(-a_{1,1}, -a_{2,1}, 0)$ , for sufficiently large  $t$  the MD-characteristic at points of the triangle attains the minima at  $(1, 0, 0)$  or at  $(a_{1,1}, a_{2,1}, 0)$ . Since  $R_{m,n}^{1,\Omega}(t)$  takes the point  $(1, 0, 0)$  to the point  $(a_{1,1}, a_{2,1}, 0)$ , the values of the MD-characteristic at  $(1, 0, 0)$  and at  $(a_{1,1}, a_{2,1}, 0)$  coincide.

Therefore, for sufficiently large  $m$  the matrix

$$H_{\langle a_{1,1}, a_{2,1} | a_{1,2}, a_{2,2}, a_{3,2} \rangle}^{(1,0,0)}(m-t, n)$$

is always reduced and there are no other reduced matrices integer congruent to the given one. We have proved the statement of Theorem 5.11 for any NRS-ray with asymptotic direction  $(-1, 0)$ .

*Step 4.2. The case of NRS-rays with asymptotic direction  $(a_{1,1}, a_{2,1})$ .* This case is similar to the case of NRS-rays with asymptotic direction  $(-1, 0)$ . We leave the details of the proof to the reader.

Proof of Theorem 5.11 is completed.  $\square$

**5.4. Examples of NRS-matrices for a given Hessenberg type.** In this subsection we bring together some examples of families  $NRS(\Omega)$  for the Hessenberg types:

$$\langle 0, 1 | 0, 0, 1 \rangle, \quad \langle 0, 1 | 1, 0, 2 \rangle, \quad \langle 0, 1 | 1, 1, 2 \rangle, \quad \langle 0, 1 | 1, 0, 3 \rangle, \quad \text{and} \quad \langle 1, 2 | 1, 1, 3 \rangle.$$

**5.4.1. Hessenberg perfect NRS-matrices  $H_{\langle 0,1|0,0,1 \rangle}^{(1,0,0)}(m, n)$ .** The Hessenberg complexity of all these matrices is one, and, therefore, they are all reduced, see the family on Figure 3 on page 18.

**5.4.2. Hessenberg perfect NRS-matrices  $H_{\langle 0,1|1,0,2 \rangle}^{(1,0,0)}(m, n)$ .** The Hessenberg complexity of these matrices equals 2. Experiments show that 12 of such matrices are nonreduced, see the family in Figure 1 on page 9. It is conjectured that all others Hessenberg matrices of  $NRS(\langle 0, 1 | 1, 0, 2 \rangle)$  are reduced.

In Figures 4, 5, and 6 the dark gray squares corresponds to nonreduced operators. We also fill with gray the squares corresponding to reduced Hessenberg matrices that are  $n$ -th powers (where  $n \geq 2$ ) of some integer matrices.

**5.4.3. Hessenberg perfect NRS-matrices  $H_{\langle 0,1|1,1,2 \rangle}^{(1,0,1)}(m, n)$ .** See the family in Figure 4. The Hessenberg complexity of these matrices equals 2. We have found 12 nonreduced matrices in the family. It is conjectured that all other Hessenberg matrices of  $NRS(\langle 0, 1 | 1, 1, 2 \rangle)$  are reduced.

**5.4.4. Hessenberg perfect NRS-matrices  $H_{\langle 0,1|1,0,3 \rangle}^{(1,0,2)}(m, n)$ .** See the family in Figure 5. The Hessenberg complexity of these matrices equals 3. We have found 6 nonreduced matrices in the family. It is conjectured that all other Hessenberg matrices of  $NRS(\langle 0, 1 | 1, 0, 3 \rangle)$  are reduced.



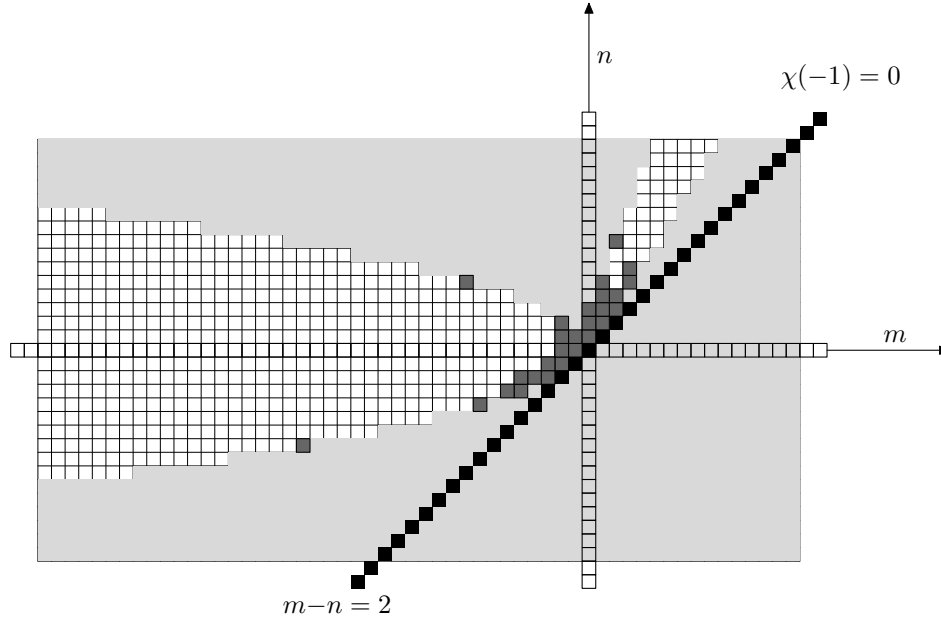


FIGURE 6. The family of Hessenberg matrices  $H_{(1,2|1,1,3)}^{(0,0,-1)}(m,n)$ .

5.4.5. *Hessenberg perfect NRS-matrices*  $H_{(1,2|1,1,3)}^{(0,0,-1)}(m,n)$ . See the family in Figure 6. This is a more complicated example of a family of Hessenberg perfect NRS-matrices, their complexity equals 12. We have found 27 nonreduced matrices in the family. It is conjectured that all other Hessenberg matrices of  $NRS(\langle 1, 2|1, 1, 3 \rangle)$  are reduced.

5.5. **Two examples of couples of Hessenberg matrices.** Let us give an example of an operator with two distinct reduced perfect Hessenberg matrices.

**Example 5.21.** The following two matrices

$$\begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 0 \\ 0 & 3 & 8 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 2 & 5 \\ 1 & 1 & 2 \\ 0 & 3 & 7 \end{pmatrix}$$

are not integer conjugate but have the same Hessenberg complexity equal to 3 and equivalent characteristic polynomials.

The following examples shows that Hessenberg complexity together with characteristic polynomial do not distinguish all the conjugacy classes.

**Example 5.22.** The matrices  $M_1$   $M_2$ , where

$$M_1 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 3 & 5 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 0 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 3 & 4 \end{pmatrix}$$

are both reduced Hessenberg matrices (with Hessenberg complexity equal to 3) of the same operator  $A$  in two different integer bases. The reason for that is the following. The sail of  $A$  in the basis of the matrix  $M_1$  contains two different integer vertices  $p_1 = (-1, 0, 0)$  and  $p_2 = (0, 1, -1)$  in a fundamental domain that are not in the same Dirichlet orbit.

## 6. OPEN PROBLEMS AND QUESTIONS

In this section we discuss several directions for further study. In Subsection 6.1 we make a brief analysis of all the observed in the previous section examples in NRS-case and formulate related questions. In Subsection 6.2 we observe the cases of three dimensional RS-matrices. Finally we say a few words about four dimensional case in Subsection 6.3.

**6.1. Three dimensional NRS-matrices.** As we have shown in Theorem 5.11 the number of nonreduced matrices in NRS-rays is always finite. Actually, from the experiments we conjecture that a stronger statement is true.

**Problem 1.** Let us fix some perfect Hessenberg type of three-dimensional matrices. Is it true that almost all (except a finite number) perfect Hessenberg NRS-matrices of a given Hessenberg type are reduced? For which Hessenberg types is the number of such matrices finite? Find the numbers in such cases.

The next two problems are actual in the cases when the number of nonreduced matrices is finite and infinite respectively.

First we show a small table of conjectured numbers of nonreduced perfect matrices for given Hessenberg types. We list all the types of Hessenberg complexity less than 5. The Hessenberg complexity of  $\Omega$  is denoted by  $\varsigma(\Omega)$ , the conjectured number of nonreduced NRS-matrices of this type is denoted by  $\#(\Omega)$ .

$\Omega$	$\langle 0, 1 0, 0, 1 \rangle$	$\langle 0, 1 1, 0, 2 \rangle$	$\langle 0, 1 1, 1, 2 \rangle$	$\langle 0, 1 1, 0, 3 \rangle$	$\langle 0, 1 1, 1, 3 \rangle$	$\langle 0, 1 1, 2, 3 \rangle$
$\varsigma(\Omega)$	1	2	2	3	3	3
$\#(\Omega)$	0	12	12	6	10	10

$\Omega$	$\langle 0, 1 2, 0, 3 \rangle$	$\langle 0, 1 2, 1, 3 \rangle$	$\langle 0, 1 2, 2, 3 \rangle$	$\langle 1, 2 0, 0, 1 \rangle$	$\langle 0, 1 1, 0, 4 \rangle$	$\langle 0, 1 1, 1, 4 \rangle$
$\varsigma(\Omega)$	3	3	3	4	4	4
$\#(\Omega)$	14	10	10	94	6	8

$\Omega$	$\langle 0, 1 1, 2, 4 \rangle$	$\langle 0, 1 1, 3, 4 \rangle$	$\langle 0, 1 3, 0, 4 \rangle$	$\langle 0, 1 3, 1, 4 \rangle$	$\langle 0, 1 3, 2, 4 \rangle$	$\langle 0, 1 3, 3, 4 \rangle$
$\varsigma(\Omega)$	4	4	4	4	4	4
$\#(\Omega)$	10	8	10	12	8	8

At this moment it is only known that there are no nonreduced matrices of type  $(\langle 0, 1|0, 0, 1 \rangle)$ .

**Problem 2.** Study the asymptotics of the number of reduced operators with respect to the growth of the Hessenberg complexity.

Consider now the family  $NRS(\langle 0, 1|1, 1, 2 \rangle)$  shown in Figure 4. Note that any operator with matrix

$$H_{\langle 0, 1|1, 1, 2 \rangle}^{(1,0,1)}(-2t^2, -2t - 1) \quad \text{for } t > 0$$

and any operator with matrix

$$H_{\langle 0, 1|1, 1, 2 \rangle}^{(1,0,1)}(2u, 2u^2 - 1) \quad \text{for } u > 0$$

is a square of some other  $SL(3, \mathbb{Z})$ -operator. All these operators are contained in two parabolas at the "boundary" of the family  $NRS(\langle 0, 1|1, 1, 2 \rangle)$ . Theoretically a similar situation can take place in the case of nonreduced operators. In the case of negative answer to Problem 1 it is interesting to know the answer to the following question.

**Problem 3.** Is it true that for any Hessenberg type the set of nonreduced perfect matrices is contained in a finite number of parabolas? How many infinite series do we have for a given Hessenberg type?

**6.2. Three dimensional RS-matrices.** We study the real spectrum case (of RS-matrices) in a particular example of the family of Hessenberg matrices  $H_{\langle 0, 1|1, 0, 2 \rangle}^{(1,0,1)}(m, n)$ :

$$H_{\langle 0, 1|1, 0, 2 \rangle}^{(1,0,1)}(m, n) = \begin{pmatrix} 0 & 1 & n+1 \\ 1 & 0 & m \\ 0 & 2 & 2n+1 \end{pmatrix}.$$

By definition, the Hessenberg complexity of the matrices of the family equals 2. Hence a Hessenberg matrix  $H_{\langle 0, 1|1, 0, 2 \rangle}^{(1,0,1)}(m, n)$  is reduced iff it is not conjugate to some matrix of the unit Hessenberg complexity, i.e., to some matrix of Hessenberg type  $\langle 0, 1|0, 0, 1 \rangle$ .

By Proposition 3.4, the matrix  $H_{\langle 0, 1|1, 0, 2 \rangle}^{(1,0,1)}(m, n)$  is reduced iff the set of absolute values of the MD-characteristic of the operator with matrix  $H_{\langle 0, 1|1, 0, 2 \rangle}^{(1,0,1)}(m, n)$  does not attain the value 1 at integer points.

In Figure 7 we study the matrices  $H_{\langle 0, 1|1, 0, 2 \rangle}^{(1,0,1)}(m, n)$  with

$$-20 \leq m, n \leq 20.$$

The square in the intersection of the  $m$ -th column and  $n$ -th row corresponds to the matrix  $H_{\langle 0, 1|1, 0, 2 \rangle}^{(1,0,1)}(m, n)$ . The square is colored in black if the characteristic polynomial has rational roots. The square is colored in gray if the matrix is irreducible and there exists an integer vector  $(x, y, z)$  with the coordinates satisfying

$$-1000 \leq x, y, z \leq 1000,$$

such that the MD-characteristic of the operator with matrix  $H_{\langle 0, 1|1, 0, 2 \rangle}^{(1,0,1)}(1, 0, 1)(m, n)$  attains the value 1 at  $(x, y, z)$ . All the rest of the squares are white.

If the square is gray, then the corresponding matrix is not reduced. If a square is white, then we cannot conclude whether the matrix is reduced or not (since the integer vector with MD-characteristic equal to 1 may have coordinates with absolute values greater than 1000).

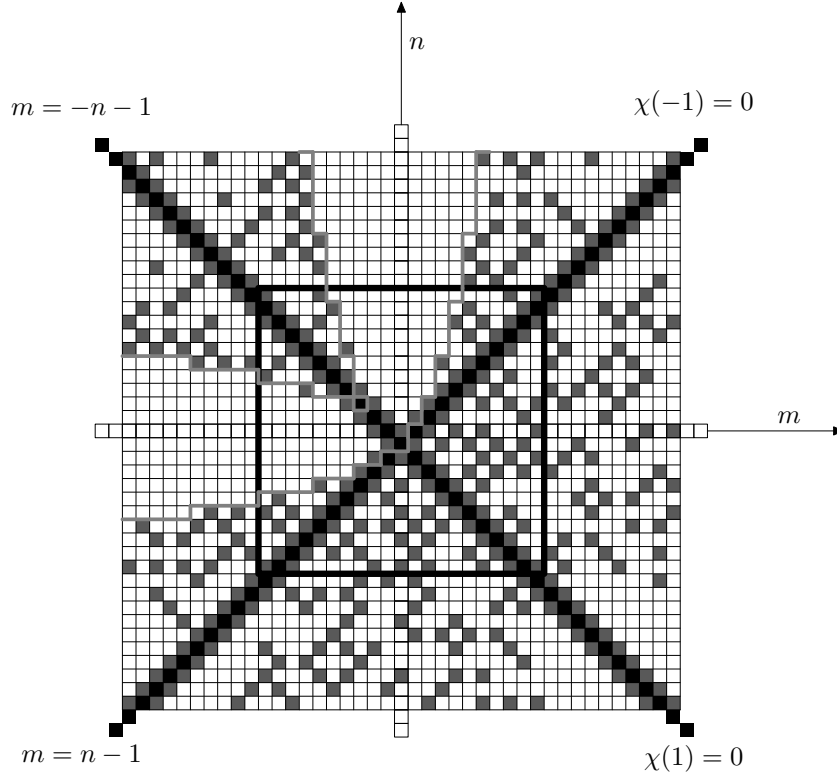


FIGURE 7. The family of matrices of Hessenberg type  $\langle 0, 1|1, 0, 2 \rangle$ .

From Figure 7 the *NRS-domain* can be easily visualized. This domain is almost completely consists of white squares. We draw a boundary broken line between the NRS-squares and the other squares with gray.

*Remark 6.1.* We have checked explicitly all the periods for all sails of the continued fractions for operators with matrices  $H_{\langle 0, 1|1, 0, 2 \rangle}^{(1, 0, 1)}(m, n)$  with

$$-10 \leq m, n \leq 10.$$

These matrices are contained inside the black square in Figure 7. It turns out that the exact result for this set completely coincides with the above approximation. If the square is gray, then the corresponding matrix is not reduced. If the square is white, then the matrix is reduced.

**Statement 6.2.** *If an integer  $m+n$  is odd, then the matrix  $H_{\langle 0, 1|1, 0, 2 \rangle}^{(1, 0, 1)}(m, n)$  is reduced.*



*Proof.* Note that the MD-characteristic at a vector  $(x, y, z)$  for the operator with Hessenberg matrix  $H_{\langle 0,1|1,0,2 \rangle}^{(1,0,1)}(m, n)$  equals

$$\begin{aligned} & | -2x^3 - (4n+2)x^2y - (4n^2+4m+4n)x^2z + (4m+2)xy^2 + \\ & (4mn+2m+6n+6)xyz + (-2m^2+4n^2+2m+6n+2)xz^2 - 2y^3 - \\ & (2m+2n+2)y^2z - (2mn+2n+2)yz^2 + (m^2-n^2-2n-1)z^3 |. \end{aligned}$$

From this expression it follows that the MD-characteristic with odd  $m+n$  takes only even values at any integer point  $(x, y, z)$ .  $\square$

*Remark 6.3.* Now we can say nothing about the asymptotic behavior of RS-matrices. From one hand Statement 6.2 implies that there exist rays of reduced operators. From the other hand all matrices corresponding to integer points of the lines

1)  $m = n$ ; 2)  $m = n + 2$ ; 3)  $m = -n$ ; 4)  $m = -n - 2$ ; 5)  $n = 3m - 4$ ; 6)  $m = 3n + 6$  are reduced (we do not state that the list of such lines is complete).

So the following problems arise. Denote by  $S_p$  the square

$$\{(m, n) \mid -p \leq m, n \leq p\}.$$

For a fixed Hessenberg type  $\Omega$  denote by  $\Theta_\Omega(p)$  the ratio of the number of reduced matrices of type  $\Omega$  corresponding to integer points of  $S_p$  to the number of all integer points of  $S_p$  (i.e., to  $(2p+1)^2$ ).

**Problem 4.** For any Hessenberg type  $\Omega$  find all limit points for the sequence  $\Theta_\Omega(p)$ ,  $p = 1, 2, 3, \dots$ . Give the upper and the lower estimates for the set of limit points of the sequence.

It is more likely that the limit exists and

$$\lim_{p \rightarrow +\infty} \Theta_\Omega(p) = 1.$$

**6.3. Four-dimensional case.** As we have already studied, the families of NRS-matrices in  $SL(3, \mathbb{Z})$  for distinct Hessenberg types has similar structures (see in the proof of Corollary 5.14). For instance, they are contained in the union of two parabolas. The reason for that is the  $SL(3, \mathbb{Q})$ -congruence of all the families with expanded integer parameters to rational parameters. In higherdimensional cases we have the same situation.

In this section we take a first glance at  $SL(4, \mathbb{Z})$ , studying a family of Hessenberg matrices of Hessenberg type  $\langle 0, 1|0, 0, 1|1, 3, 1, 4 \rangle$  of Hessenberg complexity 4. Let

$$H_{\langle 0,1|0,0,1|1,3,1,4 \rangle}^{(0,1,0,1)}(l, m, n) = \begin{pmatrix} 0 & 0 & 1 & n \\ 1 & 0 & 3 & 3n+l+1 \\ 0 & 1 & 1 & n+m \\ 0 & 0 & 4 & 4n+1 \end{pmatrix}.$$

Further we write  $H(l, m, n)$  instead of  $H_{\langle 0,1|0,0,1|1,3,1,4 \rangle}^{(0,1,0,1)}(l, m, n)$ , for short.

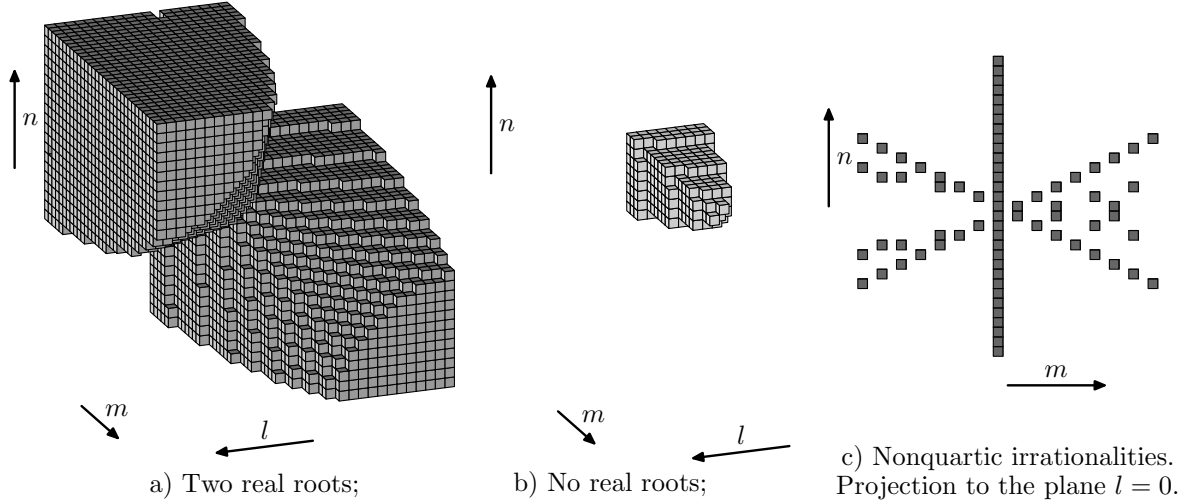


FIGURE 8. The family of matrices of the Hessenberg type  $\langle 0, 1|0, 0, 1|1, 3, 1, 4 \rangle$ .

In Figure 8 we show the family of matrices  $H(l, m, n)$  with integer parameters  $l, m, n$  satisfying

$$-15 \leq l, m, n \leq 15.$$

An operator  $H(l, m, n)$  is represented by the cube with unit edges parallel to the axes and with center at point  $(l, m, n)$ . Gray cubes of Figure 8a) correspond to the matrices with two complex and two real eigenvalues. Light-gray cubes of Figure 8b) correspond to the matrices with two distinct couples of complex-conjugate eigenvalues.

As before, we study the set of matrices with irreducible characteristic polynomial over rational numbers. So we get the following restrictions on admissible triples  $(l, m, n)$ .

First, the characteristic polynomial does not have rational roots. This is equivalent to the following: the integers 1 and  $-1$  are not the roots of the polynomial. The characteristic polynomial for the operator  $H(l, m, n)$  in  $t$  is

$$t^4 + (-4n - 2)t^3 + (-4m - 2)t^2 + (2 - 4l)t + 1.$$

Then, the number 1 (or  $-1$ ) is a root of the characteristic polynomial iff  $n+m+l = 0$  (or  $n-m-l = 0$  respectively).

Secondly, the characteristic polynomial is not a product of two polynomials of degree 2 with rational coefficients. There are exactly two series of such decomposable polynomials:

$$(t^2 + at + 1)(t^2 + bt + 1); \quad (t^2 + at - 1)(t^2 + bt - 1),$$

where  $a$  and  $b$  are integers. The characteristic polynomial of the matrix  $A(l, m, n)$  with integer parameters  $(m, l, n)$  is decomposable into two polynomials of degree 2 with rational coefficients iff at least one of the following systems has an integer solution in  $(a, b)$

variables:

$$\left\{ \begin{array}{lcl} (-b - 4l + 2)b + 2 & = & -4m - 2 \\ -b - 4l + 2 & = & a \\ l - n - 1 & = & 0 \end{array} \right. ; \quad \left\{ \begin{array}{lcl} (-b + 4l - 2)b - 2 & = & -4m - 2 \\ -b + 4l - 2 & = & a \\ l + n & = & 0 \end{array} \right. .$$

All the solutions of the above systems correspond to certain cubes with centers lying on the plane  $l - n - 1 = 0$  and  $l + n = 0$  respectively. The projection along the  $l$ -axis to the plane  $l = 0$  of all cubes corresponding to the integer solutions of the first system coincides with the projection of the cubes of integer solutions of the second system. In Figure 8c) the projections of these cubes are dark-gray squares.

We do not worry that the discriminant of the characteristic polynomial is nonzero, since it is zero only if the characteristic polynomial is reducible over rational numbers.

We conclude with several open questions.

**Problem 5.** Study an asymptotic behaviour of the set of nonreduced operators in the families of perfect Hessenberg operators in  $SL(4, \mathbb{Z})$  with fixed Hessenberg types.

The answers to Problem 5 are interesting for the subfamilies of the matrices with four real eigenvalues, with two real and two complex eigenvalues, and with four complex eigenvalues.

It is also interesting to know the answers to the following questions:

*Q.1. Is it true that there is a unique asymptotic direction for the family of Hessenberg matrices with all four non-real eigenvalues with fixed Hessenberg type?*

*Q.2. How many parameters does the family of asymptotic directions for the case of Hessenberg matrices with two real and two complex eigenvalues for a fixed Hessenberg type have?*

*Q.3. Is it true that the set of asymptotic directions of Hessenberg matrices with all real eigenvalues is everywhere dense in the space of all asymptotic directions with the natural topology?*

*Q.4. Is it true that for any Hessenberg type there exist only finitely many nonreduced Hessenberg matrices with two real and two complex eigenvalues (with four complex eigenvalues) for a fixed Hessenberg type?*

## REFERENCES

- [1] V. I. Arnold, *Continued fractions*, M.: Moscow Center of Continuous Mathematical Education, (2002).
- [2] Z. I. Borevich, I. R. Shafarevich, *Number theory*, 3 ed, M., (1985).
- [3] J. A. Buchmann, *A generalization of Voronoi's algorithm I, II*, Journal of Number Theory, v. 20(1985), pp. 177–209.
- [4] H. Davenport, *On the product of three homogeneous linear forms, I*, Proc. London Math. Soc., v. 13(1938), pp. 139–145
- [5] H. Davenport, *On the product of three homogeneous linear forms, II*, Proc. London Math. Soc.(2), v. 44(1938), 412–431.
- [6] H. Davenport, *On the product of three homogeneous linear forms, III*, Proc. London Math. Soc.(2), v. 45(1939), pp. 98–125.

- [7] H. Davenport, *Note on the product of three homogeneous linear forms*, J. London Math. Soc., v. 16(1941), pp. 98–101.
- [8] H. Davenport, *On the product of three homogeneous linear forms. IV*, Math. Proc. Cambridge Philos. Soc., v. 39(1943), pp 1–21.
- [9] C. Hermite, *Letter to C. D. J. Jacobi*, J. Reine Angew. Math. v. 40(1839), p. 286.
- [10] K. Hensel, *Thesis*, Darmstadt, Germany: Technische Hochschule, 1942.
- [11] O. Karpenkov, *On tori decompositions associated with two-dimensional continued fractions of cubic irrationalities*, Func. An. and Appl., v. 38(2004), no 2, pp. 28–37.
- [12] O. Karpenkov, *Three examples of three-dimensional continued fractions in the sense of Klein*, C. R. Acad. Sci. Paris, Ser.I, 343(2006), pp. 5–7.
- [13] O. Karpenkov, *On determination of periods of geometric continued fractions for two-dimensional algebraic hyperbolic operators*, preprint, August 2007, <http://arxiv.org/abs/0708.1604>
- [14] S. Katok, *Continued fractions, hyperbolic geometry and quadratic forms*, MASS selecta, Amer. Math. Soc., Providence, RI, (2003), pp. 121–160.
- [15] F. Klein, *Ueber eine geometrische Auffassung der gewöhnliche Kettenbruchentwicklung*, Nachr. Ges. Wiss. Göttingen Math-Phys. Kl., 3, (1895), pp. 357–359.
- [16] F. Klein, *Sur une représentation géométrique de développement en fraction continue ordinaire*, Nouv. Ann. Math. 15(3), (1896), pp. 327–331.
- [17] M. L. Kontsevich and Yu. M. Suhov, *Statistics of Klein Polyhedra and Multidimensional Continued Fractions, Pseudoperiodic topology*, Amer. Math. Soc. Transl., v. 197(2), (1999), pp. 9–27.
- [18] E. I. Korkina, *Two-dimensional continued fractions. The simplest examples*, Proceedings of V. A. Steklov Math. Ins., v. 209(1995), pp. 143–166.
- [19] G. Lachaud, *Polyèdre d'Arnold et voile d'un cône simplicial: analogues du théoreme de Lagrange*, C. R. Ac. Sci. Paris, v. 317(1993), pp. 711–716.
- [20] J. Lewis, D. Zagier, *Period functions and the Selberg zeta function for the modular group*, in The Mathematical Beauty of Physics, Adv. Series in Math. Physics 24, World Scientific, Singapore (1997), pp. 83–97.
- [21] Yu. I. Manin, M. Marcolli, *Continued fractions, modular symbols, and non-commutative geometry* (2001), <http://arxiv.org/abs/math/0102006>.
- [22] A. Markoff, *Sur les formes quadratiques binaires indéfinies*, Math. Ann., v. 15(1879), pp. 381–409.
- [23] G. F. Voronoy, *On a Generalization of the Algorithm of Continued Fractions*, Izd. Varsh. Univ., Varshava (1896); Collected Works in 3 Volumes (1952), v. 1, Izd. Akad. Nauk Ukr, SSSR, Kiev (in Russian).

*E-mail address*, Oleg Karpenkov: [karpenk@mccme.ru](mailto:karpenk@mccme.ru)

TU GRAZ /KOPERNIKUSGASSE 24, A 8010 GRAZ, AUSTRIA/